Dynamic Asset Allocation with Stochastic Income and Interest Rates[*]

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Abstract. We show how labor income affects the optimal stock/bond/cash allocation of long-term investors. We set up, calibrate, and solve a specific model with stochastic interest rates and with a stochastic labor income that can be instantaneously correlated with both interest rates, bond prices, and stock prices. The model allows the expected labor income growth rate to be an affine function of the real short-term interest rate in order to encompass business cycle variations in wages, bonuses, and lay-offs. Our calibration of the model based on PSID income data supports such a relation with a substantial variation across individuals in the business cycle sensitivity of income, i.e. the slope of this affine function. We demonstrate that this slope is crucial for the valuation and riskiness of the human capital and, consequently, for the optimal stock/bond/cash allocation both in an unconstrained complete market version of the model and in a more realistic incomplete market version with liquidity and short-sales constraints.

Keywords. Portfolio management, labor income risk, interest rate risk, intertemporal hedging, liquidity constraints, business cycle, life-cycle

JEL classification. G11, E21

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1 Introduction

It is well-documented in the theoretical asset allocation literature that the inclusion of labor income has dramatic effects on the optimal long-term portfolio choice of individual investors. Several studies, e.g. Heaton and Lucas (1997) and Cocco, Gomes, and Maenhout (2005), conclude that for an empirically reasonable insignificant correlation between labor income shocks and stock market shocks, the labor income stream is a substitute for an investment in the risk-free asset so that the financial wealth should be directed to stocks (typically, significantly levered, if possible). However, as these studies are cast in a setting where interest rates are assumed constant, they cannot distinguish short-term risk-free assets (cash deposits) from long-term risk-free assets (Treasury bonds). In order to investigate when human capital resembles a long-term bond and when it resembles cash and to assess the implications for the optimal stock/bond/cash portfolio choice, we set up, calibrate, and solve a specific model with stochastic interest rates and with a stochastic labor income that can be instantaneously correlated with both interest rates, bond prices, and stock prices.

A special and important feature of our model is that the expected labor income growth rate is an affine function of the real short-term interest rate in order to encompass business cycle variations in wages, bonuses, and lay-offs. Our calibration of the model based on PSID income data supports such a relation with a substantial variation across individuals in the business cycle sensitivity of income, i.e. the slope of the relation between expected income growth and the short-term interest rate. We demonstrate that this slope is crucial for the valuation and riskiness of the human capital and, consequently, for the optimal stock/bond/cash allocation. If the expected labor income growth is non-cyclical (zero slope), the human capital substitutes a long-term coupon bond. In that case the optimal unconstrained investment of the financial wealth involves a large long position in stocks and significant borrowing, and will typically still involve a long position in long-term bonds for speculation and intertemporal hedging purposes. If the income is counter-cyclical (negative slope), the human capital is equivalent to a levered position in a long-term bond, and a smaller [larger] share of the financial wealth should be allocated to bonds [cash]. If the income is pro-cyclical (positive slope) and the slope is exactly equal to one, the human capital will substitute for cash only. If the slope is higher than one, the human capital is like having a short position in a long-term bond and more than 100% in cash. If the slope is between zero and one, the human capital is equivalent to a moderate long position in cash and in a long-term bond. The optimal weights of the long-term bond and cash in the financial portfolio are thus highly dependent on the business cycle variations of labor income.

We also show that the introduction of liquidity constraints and short-sales constraints or significant contemporaneous correlation between the labor income and asset returns can modify the above conclusions, but the slope of the relation between income growth and the short rate remains very important for the allocation of financial wealth to bonds, stocks, and cash.

Let us briefly review the relevant literature for this study. As first noted by Merton (1971), long-term investors will generally hedge stochastic variations in the investment opportunity set. Stochastic interest rates is an important source of shifts in investment opportunities, and the effect of interest rate uncertainty on the optimal strategies of an investor without labor income is by now relatively well-studied. Sørensen (1999) and Brennan and Xia (2000) consider interest rate dynamics as in the Vasicek (1977) model and assume complete financial markets and constant market prices of both interest rate risk and stock market risk. They find that the optimal investment strategy of an investor with power utility of terminal wealth only is a simple combination of the mean-variance optimal portfolio, i.e. the optimal portfolio assuming investment opportunities do not change, and the zero-coupon bond maturing at the end of the investment horizon. Other studies of dynamic portfolio choice with uncertain interest rates include Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (2001), Deelstra, Grasselli, and Koehl (2000), Munk and Sørensen (2004), Sangvinatsos and Wachter (2005), and Liu (2007). None of these papers take into account a labor income stream of the investor, although labor income is the main source of funds for most individuals.

On the other hand, several papers discuss how the presence of a labor income process affects the consumption and investment decisions of individual investors in an environment of constant investment opportunities. A deterministic income stream is equivalent to an implicit investment in the risk-free asset and, hence, it is optimal to invest a higher fraction of financial wealth in the risky assets than in the no-income case; cf., e.g., Hakansson (1970) and Merton (1971). With stochastic income, but fully hedgeable income risk, the optimal unconstrained strategies can be deduced from the optimal strategies without labor income (Bodie, Merton, and Samuelson 1992): given the risk structure of human capital, the financial investment is determined in order to obtained the desired overall risk exposure. Since the human capital of long-term investors is often very large compared to financial wealth, labor income has dramatic effects on their optimal portfolios. Duffie, Fleming, Soner, and Zariphopoulou (1997), Koo (1998), and Munk (2000) study (mostly by use of numerical methods) the valuation of income and the optimal consumption and investment strategies of an infinite-horizon, liquidity constrained power utility investor with non-spanned income risk. The presence of liquidity constraints can significantly decrease the individual's implicit valuation of the future income stream and, hence, dampen the quantitative effects of income on portfolio choice. Other recent papers on consumption and portfolio choice with stochastic income include He and Pagès (1993), Heaton and Lucas (1997), Viceira (2001), Constantinides, Donaldson, and Mehra (2002), and Cocco, Gomes, and Maenhout (2005). Besides working with constant investment opportunities, the concrete models with stochastic income in these papers assume a single risky asset, interpreted as the stock market index. Since different risky assets will have different correlations with the labor income of a given individual, this assumption is not without loss of generality. We allow for multiple risky assets (the stock market index and bonds) and a link between labor income and investment opportunities.

Lynch and Tan (2006) consider a model where the stock market dividend yield predicts both stock market returns and the expected growth (and potentially also the volatility) of labor income leading to a negative hedging demand for stocks that partially off-sets the high speculative stock demand. We model the business cycle sensitivity of income growth through the interest rate level instead of the dividend yield. While their model apparently includes time-varying interest rates they do not allow for investments in bonds, only in cash and the stock market. Benzoni, Collin-Dufresne, and Goldstein (2007) postulate a long-run cointegration between labor income and stock market dividends and show that such a relation can substantially reduce optimal stock holdings for sufficiently risk-averse long-term investors. In contrast to these two papers, we focus on the joint implications of stochastic interest rates and labor income for the valuation of human capital and for the stock/bond/cash allocation. The model of Koijen, Nijman, and Werker (2007) includes both stochastic interest rates and labor income, but the income process is assumed to have constant expected growth and constant volatility so it cannot capture business cycle variations in income. On the other hand, they allow for time-variation in bond risk premia and inflation.

The rest of the paper is organized as follows. In Section 2 we set up the general model of the financial market, specify the preferences and income of the individual, and calibrate the model to data. Section 3 focuses on the solution of the utility maximization problem. We derive and discuss a closed-from solution to the problem with spanned income uncertainty and unconstrained investment strategies. While income is only fully spanned by traded assets for unrealistic correlations, the closed-form solution allows us to understand the economic forces at play. Furthermore, we outline the numerical solution procedure for the case with unspanned income uncertainty and relevant portfolio constraints. We present and discuss numerical results on optimal strategies in Section 4. Section 5 gives some concluding remarks. The appendices contain proofs of propositions and a detailed description of the calibration procedure and the numerical method applied for the case with unspanned income and liquidity constraints.

2 Description of the model

We model the intertemporal consumption and investment choice of a price-taking individual who can trade in stocks, bonds, and an instantaneously risk-free asset and receives a stochastic stream of income from non-financial sources, say labor income. We assume that the economy has a single perishable consumption good which serves as a numeraire so that all asset prices, interest rates, and income rates are specified in units of this good, i.e. in *real* terms.

2.1 Financial assets

We assume that the real short-term interest rate follows the Vasicek (1977) model,

$$dr_t = \kappa \left(\bar{r} - r_t \right) \, dt - \sigma_r \, dz_{rt}, \tag{1}$$

where κ, $\bar{r}$, and σ_r are positive constants, and $z_r = (z_{rt})_{t \geq 0}$ is a standard Brownian motion. The market price of interest rate risk, λ_r, is assumed constant. The price of a zero-coupon bond paying one unit of account at some time $\bar{T}$ is then given by

$$B_t^{\bar{T}} \equiv B^{\bar{T}}(r_t, t) = e^{-a(\bar{T}-t) - b(\bar{T}-t) r_t}, \tag{2}$$

where

$$b(\tau) = \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right), \tag{3}$$

$$a(\tau) = R_\infty[\tau - b(\tau)] + \frac{\sigma_r^2}{4\kappa} b(\tau)^2, \tag{4}$$

$$R_\infty = \bar{r} + \frac{\sigma_r \lambda_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}. \tag{5}$$

Here R_∞ is the limit of the yield of a zero-coupon bond as maturity goes to infinity, i.e. the asymptotic long rate.

Any desired interest rate exposure can be obtained by combining deposits/loans at the short-term interest rate (interpreted as cash or the bank account) and a single default-free real bond.[1] The dynamics of the price B_t of such a bond is given by

$$dB_t = B_t \left[\left(r_t + \sigma_B(r_t, t)\lambda_r\right) dt + \sigma_B(r_t, t)\, dz_{rt} \right], \tag{6}$$

where $\sigma_B(r_t, t) > 0$ is the bond price volatility, which generally depends on both the interest rate level and the time-to-maturity and hence on time. However, for a zero-coupon bond the volatility is $\sigma_r b(\bar{T} - t)$, which depends on the time-to-maturity $\bar{T} - t$, but not on the interest rate level. The bond price has a perfectly negative (instantaneous) correlation with the interest rate, $\rho_{Br} = -1$.

In addition to the bond, we assume that agents can invest in a single non-dividend paying stock, representing the stock market index, with price dynamics

$$dS_t = S_t \left[(r_t + \psi)\, dt + \sigma_S \left(\rho_{SB}\, dz_{rt} + \sqrt{1 - \rho_{SB}^2}\, dz_{St} \right) \right], \tag{7}$$

where $z_S = (z_{St})$ is a standard Brownian motion independent of z_r, ψ is the constant expected excess return, σ_S is the constant volatility, and $\rho_{SB} = -\rho_{Sr}$ is the constant correlation between the stock and the bond.

To simplify some of the following expressions, we introduce the vector $P_t = (B_t, S_t)^\top$ of prices of both risky assets. By combining the dynamics of B_t and S_t, we get

$$dP_t = \operatorname{diag}(P_t) \left[\left(r_t \mathbf{1} + \Sigma(r_t, t)\lambda\right) dt + \Sigma(r_t, t)\, dz_t \right], \tag{8}$$

where $z = (z_r, z_S)^\top$ and

$$\Sigma(r_t, t) = \begin{pmatrix} \sigma_B(r_t, t) & 0 \\ \sigma_S \rho_{SB} & \sigma_S \sqrt{1 - \rho_{SB}^2} \end{pmatrix}.$$

Furthermore, $\lambda = (\lambda_r, \lambda_S)^\top$ is the market price of risk vector, where

$$\lambda_S = \frac{1}{\sqrt{1 - \rho_{SB}^2}} \left(\frac{\psi}{\sigma_S} - \rho_{SB} \lambda_r \right).$$

2.2 The preferences and labor income of the individual

We assume throughout the paper that the individual has a time-additive utility function of consumption c_t and possibly terminal wealth W_T and seeks to maximize

$$\mathrm{E}\left[\int_0^T e^{-\delta t} U(c_t)\, dt + \varepsilon e^{-\delta T} U(W_T) \right],$$

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[1] We assume that real, i.e. inflation-indexed, bonds are traded. Brennan and Xia (2002), Sangvinatsos and Wachter (2005), and Koijen, Nijman, and Werker (2007) study how real interest rate risk affects optimal investments when the traded bonds are nominal.

where T is the time of death, assumed non-random, and $\varepsilon \in \{0, 1\}$ indicates whether or not the individual has utility from leaving wealth to her heirs. Throughout the paper we use a power utility function

$$U(c) = \frac{1}{1-\gamma} c^{1-\gamma},$$

where $\gamma > 0$ is the constant relative risk aversion.

We set up a model of labor income which is tractable and allows us to focus on the interaction between stochastic income and stochastic interest rates. We assume that the individual receives a continuous stream of non-negative income from non-financial sources throughout her life. The income rate at time t is denoted by y_t. We assume that y_t evolves as[2]

$$dy_t = y_t \left[(\xi_0(t) + \xi_1 r_t)\, dt + \sigma_y(t) \left\{ \rho_{yP}^\top \, dz_t + \sqrt{1 - \|\rho_{yP}\|^2}\, dz_{yt} \right\} \right], \tag{9}$$

where $z_y = (z_{yt})$ is a one-dimensional standard Brownian motion independent of z_r and z_S. The expected income growth rate is allowed to depend on the level of interest rates, reflecting the intuition that for most individuals wage increases are more frequent and larger in booming periods (high interest rates) than in recessions (low interest rates); the opposite relation may hold for individuals employed in specific industries or with specific skills.

The constant vector ρ_{yP} is defined as $(\rho_{yB}, \hat{\rho}_{yS})^\top$, where $\rho_{yB} = -\rho_{yr}$ is the instantaneous correlation between the income rate and the bond price, and $\hat{\rho}_{yS} = (\rho_{yS} - \rho_{SB}\rho_{yB})/\sqrt{1 - \rho_{SB}^2}$ where ρ_{yS} is the correlation between the income rate and the stock. If $\|\rho_{yP}\|^2 \equiv \rho_{yB}^2 + \hat{\rho}_{yS}^2 = 1$, the income rate is spanned, i.e. only sensitive to the traded risks represented by z. If that is the case, and there are no portfolio constraints, the income process can be replicated by some dynamic trading strategy of the traded assets and hence valued as a traded asset.

Note that the percentage drift $\xi_0(t) + \xi_1 r_t$ and volatility $\sigma_y(t)$ are allowed to depend on time in order to reflect the empirically relevant variations in expected income growth and uncertainty over the life of an individual, cf. e.g. Hubbard, Skinner, and Zeldes (1995) and Cocco, Gomes, and Maenhout (2005). In contrast to other studies, we allow the drift to depend on the interest rate in order to incorporate the plausible link between the expected growth in income and the overall well-being of the economy.[3] Our model does not allow for jumps in income, although the risk of lay-offs resulting in significantly lower income may affect consumption and portfolio choice, at least if the unemployment state is relatively persistent; see the discussions in Cocco, Gomes, and Maenhout (2005) and Lynch and Tan (2006). However, for many individuals the possible unemployment periods are likely to be rather short and in many countries individuals can partly insure against temporary income losses due to unemployment.

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[2] As most authors, we have modeled the income stream as an exogenously given process. Of course, in real life the individual can affect her labor income to some extent by choice of education and effort. To avoid further complications of the model we do not endogenize the labor supply decision. We refer the reader to Bodie, Merton, and Samuelson (1992) and Chan and Viceira (2000).

[3] Storesletten, Telmer, and Yaron (2004) report evidence that the volatility of income shocks is significantly higher in recessions than in economic peaks but for tractability we do not allow income volatility to depend on the interest rate level.

2.3 Model calibration

We calibrate the model of asset prices and income to U.S. data. In this section we describe the data and the calibration results. Details of the calibration exercise can be found in Appendix A. We use the Panel Study of Income Dynamics (PSID) survey of the annual income of U.S. individuals in the period 1970–1992. PSID is the largest longitudinal U.S. data set with careful information on individual labor income and individual characteristics and has also been applied in other studies of asset allocation with labor income, e.g. Cocco, Gomes, and Maenhout (2005). Since PSID offers only 23 time-series observations on a yearly basis, we also calibrate the model to quarterly U.S. aggregate income data and capital market data which span the period 1951–2003 with a total of 208 observation time points. The income data is obtained from the Personal Income and Its Disposition NIPA table of the National Economic Accounts, and the applied data is the per capita disposable personal income after personal current taxes and adjusted for personal income from financial assets. The cum-dividend stock returns are constructed using quarter-end values of the S&P 500 index over the period while the S&P 500 dividends and the CPI-index data are adopted from Shiller (2000); the updated data were downloaded from Robert Shiller's homepage (`www.econ.yale.edu/∼shiller`). All income and stock prices are in real terms (using the CPI-index as deflator). Real interest rates are constructed by subtracting an estimate of the inflation rate from the 3-month nominal interest rate. The subtracted inflation rate is obtained as the average realized inflation rate in the last four quarters relative to the same quarters one year earlier. The interest data is adopted from the estimated zero-coupon bond yields in McCulloch (1990) and McCulloch and Kwon (1993) and the yields for the period 1991 to 2003 are bootstrapped from constant maturity yields from the Federal Reserve H.15 Statistical Releases.

The applied data is illustrated in Figure 1 where the NIPA income index and the stock index are scaled so that they start out in one in 1951 (and zero for the logarithmic value). The figure also displays the yearly PSID aggregated income time series (which is scaled so that it equals the NIPA index in the second quarter of 1970). The correlation coefficient between changes in the two income time series is as high as 0.9321.

[Figure 1 about here.]

We have carried out estimations both with aggregate income data (NIPA and PSID data) and with individual income data (PSID data). In line with Cocco, Gomes, and Maenhout (2005) and Campbell and Viceira (2002) we decompose the logarithm of an individual's income into a personal idiosyncratic component and a common component; details are given in Appendix A. The estimated parameters values based on aggregate income data are displayed together with our chosen benchmark parameters for the numerical analysis in Table 1. The table presents both results based on NIPA income data for the full period 1951 to 2003 as well as for the sub-period 1970 to 1992 which is relevant for the PSID income data. The estimation for PSID data is carried out for aggregated income data within three educational groups as well as for aggregation over all households. The parameter σ_u reported in the table is volatility of the common labor income component and ρ_{uS} [ρ_{ur}] is the correlation of the common income component and the stock return [the real interest rate]. For the PSID estimations we have also in Panel B of Table 1 tabulated the results for the individual household income volatility and correlations derived as explained in Appendix A.

[Table 1 about here.]

In all estimations based on PSID income data, we have fixed the parameters describing the stock dynamics and the interest rate dynamics at the values estimated for the full sample period 1951 to 2003 with quarterly NIPA income data. These parameter estimates thus reflect all the information about dynamics of stock and interest rates which is available in our capital market data. The estimations based on NIPA income data for the sub-period 1970 and 1992 indicate that the other parameters are not affected much by fixing these stock and interest rate parameters. Except for the interest rate volatility parameter, the stock and interest rate parameters are not significantly different for this sub-period than the case estimated for the full sample period.[4] The sub-period 1970 to 1992, which is relevant for the PSID data, contains the period of the so-called monetary experiment in 1978 to 1982, and the interest rate volatility is estimated significantly higher at 3.02% for the PSID sub-period compared to the full sample period estimate of 2.18%.

Our benchmark parameters are chosen close to the estimated values. Some parameters are chosen to be similar to those applied by, e.g., Cocco, Gomes, and Maenhout (2005) and Campbell and Viceira (2002) who use a stock market risk premium of $\psi = 4\%$ and a real interest rate level of 2%. The volatility in the aggregate income processes are in all cases estimated around 2%, but our benchmark parameter values is set significantly higher at 20%. This reflects the calibrated values in Panel B of Table 1, which indicate that individual household income volatility might be as high as 33–36%. These calibrated estimates are consistent with results reported by Cocco, Gomes, and Maenhout (2005) who show, however, that some of this income volatility can be attributed to transitory income effects, and therefore we have adjusted the income volatility down to 20% in our benchmark parameter set. Also consistent with the analysis in Cocco, Gomes, and Maenhout (2005), the correlation coefficients between individual household income and the stock index are all calibrated close to zero due to the large idiosyncratic component in individual household income. The correlations between household income and the real interest rate are likewise calibrated close to zero.

It may be noted that estimations based on NIPA and PSID income, respectively, provide different estimates of the income drift parameters ξ_0 and ξ_1 although the income series are highly correlated for the sub-period 1970 to 1992. The NIPA income estimations, e.g., indicate a significant real growth in income of 1.81% per year for the full period. On the other hand, the PSID estimation indicate a clear positive relation between income growth and the level of the real interest rate. This positive relation tends to increase with the educational level and is especially significant for households where the household head has a college education.

In our main analysis we fix the expected income growth parameter ξ_0 to be time-independent. For that analysis we assume that ξ_0 is equal to 2.5% so that the total expected income growth when the real interest rate is at its long-term average is $\xi_0 + \xi_1 \bar{r} = 3\%$. This reflects a young individual's expected increase in real income of about 1–2% due to common real income growth (as reflected in the estimated value of ξ_0)) and about a 1–2% expected increase in salary due getting working experience (as reflected in the slope of the life-cycle income profiles in Figure 2

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[4] A likelihood ratio test for the hypothesis that the five parameters ψ, κ, $\bar{r}$, σ_S, and ρ_{Sr} are equal to the estimates for the full period has a p-value of 75%. Including the interest rate volatility as a sixth parameter in the test results in a p-value of only 0.1%.

for younger investors). Appendix A discusses how to take life-cycle variations in labor income into account in the calibration, and the implications for portfolio choice are studied in Section 4.3.

Our estimation approach does not involve the risk premium on real interest rate risk, and our benchmark parameter value for the numerical experiments is set at $\lambda_r = 0$. We do not have reliable data on the return on real bonds to empirically justify this choice (since real bonds were first introduced in the U.S. in 1998). But the smaller risk premium on bonds relatively to stocks is at least consistent with the similar results for *nominal* bonds. For example, the Brealy, Myers, and Allen (2006) textbook reports an 1.2% (7.6%) average excess return on government bonds (stocks) with a historical standard deviation on returns of 8.2% (20.1%) over the period 1900–2003 (updates of the estimated values in Dimson, Marsh, and Staunton (2002)). In our interest data the average excess return on a 5-year zero-coupon bond is likewise 1.32% (2.35%) with a volatility of 9.40% (12.52%) for the full period 1951 to 2003 (the sub-period 1970 to 1992). Together with a zero interest rate risk premium, our benchmark parameters imply a long-term real yield of 1.92% and for an initial real short rate identical to the long-term average of 2% the real yield curve is almost flat.

In Table 2 we have tabulated estimation results obtained for individual household time series in the PSID data base. Here we only consider households with at least 10 consecutive income observations for an age below 65 years of the household head and with a household income in all years (all non-consecutive observations are excluded) of at least \$1000. These requirements reduce the number of observations from about 80000 for 7300 households to a total of 40497 observations for 2302 different households. In all estimations the stock and interest rate parameters have been fixed at the values also applied in Table 1. The results in Table 2 aim at illustrating the degree of dispersion of the obtained estimates for individual households by reporting the 10% fractile, the median, and the 90% fractile for the obtained estimates for each of the income parameters. The table also reports the median standard deviation for the different parameter estimates.

Note that especially the drift parameters ξ_0 and ξ_1 display a high degree of dispersion across individuals, and these parameters likewise seem to be estimated with much less precision than in the case for aggregate income in Table 1. This is due to the much higher volatility in the individual household time-series. In all cases in Table 2, however, the median estimates are quite close to our applied benchmark parameter values.

[Table 2 about here.]

2.4 Optimal strategies

The individual has to choose a consumption strategy $c = (c_t)$ and an investment strategy $\theta = (\theta_t)$. Here c_t is the rate at which goods are consumed at time t with the natural requirement that $c_t \geq 0$ at all times and in all states of the economy. Furthermore, θ_t is a vector $(\theta_{Bt}, \theta_{St})^\top$ of the amounts (i.e. units of the consumption good) invested at time t in the bond and the stock. With W_t denoting the financial wealth of the investor at time t, the amount invested in the bank account (held in "cash") is residually determined as $\theta_{0t} = W_t - \theta_{Bt} - \theta_{St}$. Given a consumption strategy c and an investment strategy θ, the financial wealth of the individual W_t evolves as

$$dW_t = (r_t W_t + \theta_t^\top \Sigma(r_t, t)\lambda - c_t + y_t)\, dt + \theta_t^\top \Sigma(r_t, t)\, dz_t. \tag{10}$$

The consumption and investment strategies must satisfy some technical conditions for the wealth process to be well-defined. If there are no other restrictions on the strategies (except $c_t \geq 0$) we denote by $\mathcal{A}_t^{\text{unc}}$ the set of admissible consumption and investment strategies (c, θ) over the time interval $[t, T]$. We will also consider the case where it is not possible for the individual to borrow funds using future income as collateral so that her financial wealth W_t must stay non-negative at all times and in all states of the world.[5] In a continuous-time setting this is implemented by requiring that whenever the financial wealth hits zero, the investor must eliminate her positions in bonds and stocks. After she have received labor income, she may again enter the markets for risky securities. We denote the set of admissible strategies with this constraint by $\mathcal{A}_t^{\text{con}}$.

The indirect utility function of the individual is defined as

$$J(W, r, y, t) = \sup_{(c,\theta)\in\mathcal{A}_t} \mathrm{E}_t \left[\int_t^T e^{-\delta(s-t)} U(c_s)\, ds + \varepsilon e^{-\delta(T-t)} U(W_T) \right], \tag{11}$$

where the expectation is computed given the values of W, r, y at time t and given the strategy (c, θ). The set $\mathcal{A}_t$ is either equal to $\mathcal{A}_t^{\text{unc}}$ or $\mathcal{A}_t^{\text{con}}$. With the assumed CRRA utility function the marginal utility is infinite at zero consumption so that the non-negativity constraint on consumption is not binding. The Hamilton-Jacobi-Bellman (HJB) equation associated with this dynamic optimization problem is

$$\begin{aligned}\delta J = \sup_{c,\theta} \Big\{ & U(c) + J_t + J_W \left(rW + \theta^\top \Sigma \lambda - c + y \right) + \frac{1}{2} J_{WW} \theta^\top \Sigma \Sigma^\top \theta \\ & + J_r \kappa [\bar{r} - r] + \frac{1}{2} J_{rr} \sigma_r^2 + J_y y (\xi_0 + \xi_1 r) + \frac{1}{2} J_{yy} y^2 \sigma_y^2 \\ & - J_{Wr} \theta^\top \Sigma \mathbf{e}_1 \sigma_r + J_{Wy} y \sigma_y \theta^\top \Sigma \rho_{yP} + J_{ry} y \rho_{yr} \sigma_y \sigma_r \Big\},\end{aligned} \tag{12}$$

where $\mathbf{e}_1 = (1, 0)^\top$, subscripts on J denote partial derivatives, and we have suppressed the arguments of the functions for notational simplicity. The terminal condition is $J(W, r, y, T) = \varepsilon U(W) = \varepsilon W^{1-\gamma}/(1-\gamma)$.

The first-order condition for consumption is the standard envelope condition

$$U'(c_t) = J_W(W_t, r_t, y_t, t) \quad \Rightarrow \quad c_t = \left[J_W(W_t, r_t, y_t, t) \right]^{-1/\gamma}. \tag{13}$$

For the unrestricted investment case the first-order condition for the portfolio θ implies that

$$\theta_t = -\frac{J_W}{J_{WW}} \left(\Sigma(r_t, t)^\top \right)^{-1} \lambda - \frac{J_{Wy}}{J_{WW}} y \sigma_y(t) \left(\Sigma(r_t, t)^\top \right)^{-1} \rho_{yP} + \frac{J_{Wr}}{J_{WW}} \frac{\sigma_r}{\sigma_B(r_t, t)} \mathbf{e}_1. \tag{14}$$

The first part corresponds to the standard mean-variance optimal portfolio, the second part is a hedge against changes in the income rate, while the third part is a hedge against changes in the interest rate. The income hedge term reflects a position in the portfolio with relative weights given by $\left(\Sigma(r_t, t)^\top \right)^{-1} \rho_{yP} / \mathbf{1}_{n+1}^\top \left(\Sigma(r_t, t)^\top \right)^{-1} \rho_{yP}$. This is the portfolio with the maximal absolute correlation with the income rate of the individual, cf. Ingersoll (1987, Ch. 13). This maximal correlation equals $\|\rho_{yP}\|$ so if the income rate is spanned, this correlation will equal 1. Since the

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[5] This "hard" borrowing constraint is standard in the literature. A recent paper by Davis, Kubler, and Willen (2005) studies the portfolio choice under a "soft" borrowing constraint that allows individuals to borrow even with a negative current wealth although at a rate higher than the risk-free interest rate. Their study assumes constant interest rates. To focus on the interaction between stochastic interest rates and stochastic labor income we stick to the "hard" borrowing constraint, which is easier to handle.

bond price is perfectly negatively correlated with the interest rate, the interest rate is hedged by a position in the bond only. In contrast, the income hedge and the mean-variance terms generally involve all risky assets. The remaining wealth, $W_t - \theta_t^{\top} \mathbf{1}_{n+1}$, is invested in the bank account.

3 Solutions to the optimal consumption-investment problem

In this section we solve the utility maximization problem formulated above. First, we derive a closed-from solution to the problem with spanned income uncertainty and unconstrained investment strategies. Next, we discuss the solution in the setting with unspanned income uncertainty and liquidity constraints.

3.1 Spanned income and no investment constraints

Assume that the income stream is fully spanned by the traded assets, i.e. that $\|\rho_{yP}\| = 1$, which implies that

$$\rho_{yS} + \rho_{SB}\rho_{yr} = \pm\sqrt{(1-\rho_{yr}^2)(1-\rho_{SB}^2)}. \tag{15}$$

The correlation values satisfying this condition are far from the benchmark parameters determined from our calibrations, cf. Table 1. However, with the additional assumption of no portfolio constraints, we can obtain a closed-form solution to the consumption-investment problem, which allows for a detailed understanding of various effects and also can serve as a check of the accuracy of the numerical solution technique applied for the general case.

Since the income process is spanned, any unconstrained investor can replicate it, and it can be valued as the dividend stream from a traded asset. The market value at time t of the income stream over the time period $[t, T]$ is

$$H(y, r, t) = \mathrm{E}_t^{\mathbb{Q}}\left[\int_t^T y_s e^{-\int_t^s r_v\, dv}\, ds\right], \tag{16}$$

where $\mathbb{Q}$ denotes the unique risk-neutral probability measure. We can think of the individual selling the remaining income stream for the amount $H(y, r, t)$, her human capital. As described below the optimal strategies can in this case be derived from the optimal strategies for the case without income but with a financial wealth of $W_t + H(y_t, r_t, t)$ instead of just W_t. Under our assumptions on the dynamics of the labor income rate and the short-term interest rate, we are able to derive an explicit expression of the human capital as shown in the following proposition. The proof is given in Appendix B.[6]

Proposition 1 *Under the assumptions above, the human capital is given by*

$$H(y, r, t) = yM(r, t) \equiv y \int_t^T h(t, s) \left(B^s(r, t)\right)^{1-\xi_1}\, ds, \tag{17}$$

---

[6] Given (2), the human wealth expression in (17) can also be written in the exponential-affine form $H(y, r, t) = \int_t^T \exp\{A_0(t, s) + A_1(t, s)r + A_2(t, s)\ln y\}\, ds$. This is the case in any setting where the risk-neutral dynamics of r_t and $\ln y_t$ are affine; c.f., e.g., Duffie, Pan, and Singleton (2000). We focus on a non-trivial case where the functions A_0, A_1, A_2 can be stated in closed form (involving some simple integrals).

where

$$\begin{aligned}\ln h(t,s) = &\int_t^s \big(\xi_0(u) - \sigma_y(u)\rho_{yP}^\top\lambda - (\xi_1 - 1)\rho_{yB}\sigma_r\sigma_y(u)b(s-u)\big)\,du \\ &+ \xi_1(\xi_1 - 1)\frac{\sigma_r^2}{2\kappa^2}\Big(s - t - b(s-t) - \frac{\kappa}{2}b(s-t)^2\Big)\end{aligned} \tag{18}$$

with the function b given by (3)*.*

Due to the structure of the assumed income rate process, the human capital is separated as the product of the current income rate, y, and a multiplier, $M(r,t)$, depending only on the interest rate and time. The risk characteristics of human capital will be discussed below together with the consequences for optimal portfolio choice.

With time-additive CRRA utility it is well-known that indirect utility function with wealth W_t and no income is of the form

$$V(W,r,t) = \frac{1}{1-\gamma}g(r,t)^\gamma W^{1-\gamma},$$

where $g(r,t)$ is a function that depends on the remaining investment horizon (and hence on time) and the risk aversion parameter γ; see, e.g., Ingersoll (1987, Ch. 13). With Vasicek interest rate dynamics, the function $g(r,t)$ can be computed explicitly, cf. Sørensen (1999). With a spanned income rate and no portfolio constraints, we can think of the individual having an initial financial wealth of $W_t + H(y_t, r_t, t)$ and no labor income instead of having initial wealth W_t and the income stream.[7] Under the assumptions of this section, we therefore have that the indirect utility function *with* labor income is given by

$$J(W,r,y,t) = V(W + H(y,r,t), r, t). \tag{19}$$

From the value function the optimal consumption and investment strategies can be derived from (13) and (14). The following proposition summarizes the solution, which can be verified by substitution of (20) into the HJB-equation (12).

Proposition 2 *Under the assumptions stated above, the indirect utility function is given by*

$$J(W,r,y,t) = \frac{1}{1-\gamma}g(r,t)^\gamma\left(W + H(y,r,t)\right)^{1-\gamma}, \tag{20}$$

where the function $g(r,t)$ is defined by

$$g(r,t) = \int_t^T f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds + \varepsilon f(T-t)\left(B^T(r,t)\right)^{\frac{\gamma-1}{\gamma}}$$

with $f(\tau)$ defined by

$$\ln f(\tau) = \left(-\frac{\delta}{\gamma} + \frac{1-\gamma}{2\gamma^2}\|\lambda\|^2\right)\tau + \frac{1-\gamma}{\gamma^2}\left(\left(\bar{r} - R_\infty\right)\left(\tau - b(\tau)\right) - \frac{\sigma_r^2}{4\kappa}b(\tau)^2\right).$$

The optimal consumption rate is

$$c_t = \frac{W_t + H(y_t, r_t, t)}{g(r_t, t)}, \tag{21}$$

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[7] Bodie, Merton, and Samuelson (1992) apply this idea in the case of constant investment opportunities.

while the optimal investments in the bond and the stock are

$$\begin{aligned}\theta_{Bt} &= \frac{1}{\gamma\sigma_B}\left(W_t + H_t\right)\left(\lambda_r - \rho_{SB}\frac{\lambda_S}{\sqrt{1-\rho_{SB}^2}}\right) - H\frac{\sigma_y(t)}{\sigma_B}\frac{\rho_{yB} - \rho_{yS}\rho_{SB}}{1-\rho_{SB}^2} \qquad (22)\\ &\quad + (\xi_1 - 1)\frac{\sigma_r}{\sigma_B} y \int_t^T b(s-t)h(t,s)\left(B^s\right)^{1-\xi_1}\, ds + \left(1 - \frac{1}{\gamma}\right)\frac{\sigma_r}{\sigma_B}\left(W_t + H_t\right) G(r_t, t),\\ \theta_{St} &= \frac{1}{\gamma}(W + H)\frac{\lambda_S}{\sigma_S\sqrt{1-\rho_{SB}^2}} - H\sigma_y(t)\frac{\rho_{yS} - \rho_{SB}\rho_{yB}}{\sigma_S(1-\rho_{SB}^2)}, \qquad (23)\end{aligned}$$

where

$$G(r,t) \equiv \frac{\gamma}{1-\gamma}\frac{g_r(r,t)}{g(r,t)} = \frac{\int_t^T b(s-t)f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}} + \varepsilon b(T-t)f(T-t)\left(B^T(r,t)\right)^{\frac{\gamma-1}{\gamma}}}{\int_t^T f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\, ds + \varepsilon f(T-t)\left(B^T(r,t)\right)^{\frac{\gamma-1}{\gamma}}}. \qquad (24)$$

The terms in the expression for θ_t that involve H compensate exactly for the dynamics of the human capital, since by Itô's Lemma

$$dH(y_t, r_t, t) = \dots\, dt + H_y(y_t, r_t, t) y_t \sigma_y(t)\rho_{yP}^\top\, dz_t - H_r(y_t, r_t, t)\sigma_r\, dz_{rt}.$$

The percentage volatility vector of the total wealth $W_t + H(y_t, r_t, r)$ is therefore

$$\frac{1}{\gamma}\lambda - \frac{g_r(r_t,t)}{g(r_t,t)}\sigma_r \mathbf{e}_1,$$

just as for the case without income. The intuition is that a CRRA investor will determine her investment strategy in order to obtain a given percentage volatility vector of total wealth. This desired volatility vector is independent of how the total wealth is comprised by financial wealth and human capital. In other words, the consumer-investor computes her optimal investment of total wealth and then corrects the investment strategy for the implicit investment that the income stream represents.

In the problem without labor income, the optimal strategies are such that wealth stays positive with probability one. By analogy, the optimal strategies for the problem with labor income ensure that *total* wealth stays positive with probability one. However, financial wealth in itself may very well go negative in some situations. For positive values of financial wealth it makes sense to talk of portfolio weights, i.e. the fractions of financial wealth invested in the different assets. Investment strategies are usually stated in such portfolio weights. We denote portfolio weights by $\pi = \theta/W$. Using (17), we get from (22) and (23) that the optimal portfolio weights can be written compactly as

$$\begin{aligned}\pi_t &= \frac{1}{\gamma}\left(1 + \frac{y_t}{W_t}M(r_t,t)\right)\left(\Sigma(r_t,t)^\top\right)^{-1}\left(\lambda - \gamma\sigma_y(t)\rho_{yP}\right) + \sigma_y(t)\left(\Sigma(r_t,t)\right)^{-1}\rho_{yP}\\ &\quad + \left(\frac{y_t}{W_t}M_r(r_t,t) - \frac{g_r(r_t,t)}{g(r_t,t)}\left(1 + \frac{y_t}{W_t}M(r_t,t)\right)\right)\frac{\sigma_r}{\sigma_B(r_t,t)}\mathbf{e}_1.\end{aligned} \qquad (25)$$

It is now clear that the optimal portfolio weights do not depend on current financial wealth and labor income separately but only through the wealth-to-income ratio.

Investment in the stock. The first term in θ_{St} is the speculative demand and the second term is the correction term for stock-like income risk. The presence of labor income magnifies the optimal investment in the different stocks due to a wealth effect. The sign of the correction term depends on the correlation structure. If the stock is uncorrelated with the bond, the hedge demand is positive [negative] if the income-stock correlation is negative [positive]. The total effect of income on the demand of the stock depends on the sign of $\lambda_S/\gamma - \sigma_y(t)\hat{\rho}_{yS}$. Since H is increasing in T, this sign will also determine how the optimal stock demand varies with the investment horizon. For stocks in positive demand the popular advice to decrease the fraction of wealth invested in stocks over the life-cycle so that θ_{St} increases with T is true for a risk-free income stream, but with income uncertainty the validity of this advice is highly dependent on risk aversion, risk premia, and the correlation coefficients.

Investment in the bond. The optimal bond investment in (22) is the sum of a speculative demand (first term), a correction for bond-like income risk (second and third term), and a hedge against interest rate risk (last term). Both the speculative term and the hedge term are magnified due to the presence of human capital. The first correction for bond-like income risk shows up if the contemporaneous "stock-filtered" correlation between the income rate and the short-term interest rate is non-zero. The second correction term can be rewritten as $\sigma_r \frac{\partial H}{\partial r}/\sigma_B$, where

$$\frac{\partial H}{\partial r} = (\xi_1 - 1)y_t \int_t^T h(t,s)b(s-t)\left(B^s(r_t,t)\right)^{1-\xi_1}\, ds$$

is the interest rate sensitivity of the human capital. The parameter ξ_1, i.e. the slope of the relation between expected income growth and the short rate, is crucial for the risk characteristics of the human capital and, consequently, for the optimal bond investment. For the case $\xi_1 = 1$, the human capital is insensitive to variations in the interest rate. An increase in the discount rate will be exactly offset by an increase in the expected future income rates. Ignoring contemporaneous income-asset correlations, the human capital will be equivalent to a short-term risk-free asset (cash) if $\xi_1 = 1$. Since in that cash there is no long-term bond investment implicit in the human capital, there is no correction in the optimal bond demand. For $\xi_1 = 0$ (non-cyclical labor income), the human capital is like a long-term bond paying a coupon of $h(t,s)$ at time s and, hence, the explicit bond investment is reduced. For ξ_1 between 0 and 1, the interest rate sensitivity of the human capital is equivalent to that of a portfolio of a long position in cash and a long position in the long-term coupon bond. For $\xi_1 < 0$, i.e. a counter-cyclical income, the human capital is like a levered position in the long-term bond. For $\xi_1 > 1$ (strongly pro-cyclical income), the interest rate risk of the human capital is equivalent to that of a portfolio of more than 100% in cash and a short position in the long-term bond. The explicit bond investment thus have to be corrected upwards. Recall that, based on our calibration, the benchmark value of ξ_1 is 0.25, but with a substantial variation across individuals.

In the case with no income, the second and third terms on the right-hand side of (22) disappear and $H = 0$ in the speculative term and the hedge term. In Appendix C we show that $G(r,t)$ is increasing in T. The optimal bond demand of a conservative investor with no labor income is increasing in the investment horizon. This is inconsistent with the traditional advice of investing more in bonds as the horizon shrinks. As discussed by Munk and Sørensen (2004), the hedge position in the traded bond is combined with a short-term deposit or loan to mimic an investment in a specific coupon bond reflecting the expected consumption stream of the investor. Other things equal, a longer horizon will increase the duration and volatility of this desired coupon bond, which requires a larger weight on the traded bond in the mimicking strategy. The presence of labor income can either reinforce, dampen, or reverse the horizon effect. In Appendix C we also show that $G(r,t)$ is decreasing in r if $\gamma > 1$ and increasing in r if $\gamma < 1$. Keeping the bond volatility fixed, it follows that in absence of labor income, the optimal bond allocation is a decreasing function of the interest rate level. Due to the interest rate sensitivity of human capital, the optimal bond allocation with income may respond very differently to interest rate changes.

Consumption. Using (21), we see that the propensity to consume out of wealth and the propensity to consume out of current income, respectively, are given by

$$\frac{c_t}{W_t} = \frac{1 + \frac{y_t}{W_t} M(r_t,t)}{g(r_t,t)}, \qquad \frac{c_t}{y_t} = \frac{\frac{W_t}{y_t} + M(r_t,t)}{g(r_t,t)}, \tag{26}$$

which also depend on the wealth-income ratio. As expected, the propensity to consume out of income, c_t/y_t, is increasing in the wealth-income ratio and the expected income growth rate and decreasing in the current income rate, while the dependence on the income volatility, the risk aversion coefficient, the investment horizon, and the interest rate level is parameter specific.

3.2 Unspanned income uncertainty and liquidity constraints

Now consider the case where $\|\rho_{yP}\| < 1$ implying that the income uncertainty is unspanned, i.e. not perfectly hedgeable. In addition we impose a liquidity constraint on the individual so that the financial wealth has to stay non-negative at all points in time. We will also consider the effects of imposing stricter constraints so that the investor is restricted to non-negative positions in both the bond, the stock, and the bank account. Due to the unspanned income and the investment constraints, it is no longer possible to derive a closed-form solution to the utility maximization problem, so we have to resort to numerical solution techniques.

Our numerical method is based on a finite difference backwards iterative solution of the HJB equation with an optimization over feasible consumption rates and portfolios at each (time, state) node in the lattice. The original formulation of the problem has three state variables (financial wealth, interest rate, and income rate). In order to simplify the implementation of the numerical solution algorithm, we will use a homogeneity property to reduce the number of state variable by one. It follows from the linearity of the wealth dynamics in (10) and the income dynamics in (9) that if the consumption and investment strategy (c,θ) is optimal with initial wealth and income (W,y), then $(kc,k\theta)$ is optimal with initial wealth and income (kW,ky). The assumed power utility function implies that the value function is homogeneous of degree $1-\gamma$ in (W,y), i.e.

$$J(kW,r,ky,t) = k^{1-\gamma} J(W,r,y,t),$$

from which it follows that[8]

$$J(W,r,y,t) = y^{1-\gamma} F\left(\frac{W}{y}, r, t\right), \tag{27}$$

---

[8] Of course, this is also true in the complete market case studied previously. There we have

$$F\left(\frac{W}{y}, r, t\right) = \frac{1}{1-\gamma} g(r,t)^{\gamma} \left(\frac{W}{y} + M(r,t)\right)^{1-\gamma},$$

where $M(r,t)$ is given in Proposition 1.

where we have defined $F(x,r,t) = J(x,r,1,t)$. By substitution of (27) into the HJB equation (12) we get that $F = F(x,r,t)$ solves the non-linear partial differential equation (PDE)

$$
\begin{aligned}
\hat{\delta}(r,t)F = \sup_{\hat{c},\pi} \Bigg\{ & \frac{\hat{c}^{1-\gamma}}{1-\gamma} + F_t + F_r\left(\kappa[\bar{r}-r] + (1-\gamma)\rho_{yr}\sigma_y(t)\sigma_r\right) \\
& + F_x\left(1 - \hat{c} + x\left[(1-\xi_1)r - \xi_0(t) + \gamma\sigma_y(t)^2 + \pi^\top\Sigma\left(\lambda - \gamma\sigma_y(t)\rho_{yP}\right)\right]\right) \\
& + \frac{1}{2}x^2 F_{xx}\left(\pi^\top\Sigma\Sigma^\top\pi + \sigma_y(t)^2 - 2\sigma_y(t)\pi^\top\Sigma\rho_{yP}\right) \\
& + \frac{1}{2}\sigma_r^2 F_{rr} - xF_{xr}\sigma_r\left(\pi^\top\Sigma\mathbf{e}_1 + \rho_{yr}\sigma_y(t)\right)\Bigg\},
\end{aligned}
\tag{28}
$$

where we have introduced

$$
\hat{\delta}(r,t) = \delta - (1-\gamma)(\xi_0(t) + \xi_1 r) + \frac{1}{2}\gamma(1-\gamma)\sigma_y(t)^2
$$

and $\hat{c}_t = c_t/y_t$ is the consumption-to-income ratio and $\pi_t = \theta_t/W_t$ is the vector of portfolio weights. The terminal condition on F is $F(x,r,T) = \varepsilon x^{1-\gamma}/(1-\gamma)$.

We set up a lattice in (x,r,t) and solve the PDE (28) numerically using a backward iterative procedure starting from the terminal date T. At each time t_n in the lattice we first guess on the optimal controls $\hat{c}(x_i,r_j,t_n), \pi(x_i,r_j,t_n)$ and solve (28) using finite difference techniques for $F(x_i,r_j,t_n)$, which is then a guess on the value function at time t_n. Using that in the first-order conditions for the maximization in (28), we can derive a new guess on the optimal controls, which can again be used to find a new guess on the value function. We continue these iterations until the guess on the value function at t_n is stable, and we can then move on to the previous time step t_{n-1}. Our solution technique is basically the same as that applied by Brennan, Schwartz, and Lagnado (1997) and is closely related to the well-documented Markov Chain Approximation Approach, which has previously been used to study various consumption/investment problems, cf. Fitzpatrick and Fleming (1991), Hindy, Huang, and Zhu (1997), and Munk (1999, 2000). Details on the numerical procedure can be found in Appendix D.

We ensure that financial wealth W, and hence the wealth-income ratio $x = W/y$, stay non-negative by restricting the individual's choice of consumption and portfolio whenever $x = 0$ to a zero investment in the risky assets and to a consumption level which is smaller than the current income, i.e. $\hat{c} \leq 1$. If we do not further restrict the consumption and portfolio choice, the optimal choice for a strictly positive values of x is given by the first-order conditions from (28), i.e.[9]

$$
\hat{c}_t = \left(F_x\right)^{-1/\gamma}, \tag{29}
$$

$$
\pi_t = -\frac{F_x}{xF_{xx}}\left(\Sigma^\top\right)^{-1}\left(\lambda - \gamma\sigma_y(t)\rho_{yP}\right) + \frac{F_{xr}}{xF_{xx}}\sigma_r\left(\Sigma^\top\right)^{-1}\mathbf{e}_1 + \sigma_y(t)\left(\Sigma^\top\right)^{-1}\rho_{yP}. \tag{30}
$$

Imposing the liquidity constraint makes lower levels of financial wealth worse. Without the liquidity constraint, a consumer-investor with a large human capital is not that concerned with a fall in financial wealth from a low level to zero, in fact the financial wealth can go negative. With the liquidity constraint, the consequences of losing financial wealth from a low level are more severe. If you end up at a zero financial wealth, you have to stay away from the risky assets and keep

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[9] Note that if we substitute the expression for F in the complete market case given in Footnote 8 into (30), we get (25).

consumption below current labor income. You can only return to strictly positive wealth and risky positions by consuming strictly less than your income. We therefore expect significantly less risky positions at near-zero financial wealth levels relative to the case without the liquidity constraint.

Imposing a strict no borrowing condition for all wealth levels, we must have $\pi_{Bt} + \pi_{St} \leq 1$. Of course, if the portfolio given by (30) satisfies this condition, it is still the optimal portfolio, but if the constraint is binding, we maximize in (28) over portfolios (π_{Bt}, π_{St}) with $\pi_{Bt} + \pi_{St} = 1$ and get

$$\pi_{Bt} = \frac{1}{\sigma_B^2 + \sigma_S^2 - 2\rho_{SB}\sigma_S\sigma_B} \left\{ -\frac{F_x}{xF_{xx}} \left(\sigma_B[\lambda_r - \gamma\sigma_y(t)\rho_{yB}] - \psi + \gamma\sigma_S\sigma_y(t)\rho_{yS}\right) \right. \\ \left. + \frac{F_{xr}}{xF_{xx}} (\sigma_B - \sigma_S\rho_{SB}) + \sigma_S^2 - \rho_{SB}\sigma_S\sigma_B + \sigma_y(t)[\sigma_B\rho_{yB} - \sigma_S\rho_{yS}] \right\}$$

and, of course, $\pi_{St} = 1 - \pi_{Bt}$. In a similar manner we can impose non-negativity constraints on the portfolio weights.

4 Numerical results

In the following we use the benchmark parameter values listed in Table 1 and assume that the current short-term interest rate is equal to the long-term level of 2% unless otherwise mentioned. In this section we assume constant income parameters; life-cycle variations in income will be studied in Section 4.3. Unless otherwise mentioned we consider an investor with time preference rate $\delta = 0.03$, relative risk aversion $\gamma = 4$, time horizon $T = 30$ years, and utility from both intermediate consumption and terminal wealth. The bond which the investor trades in is assumed to be a 10-year (real) zero-coupon bond.

4.1 Spanned income, no constraints

For now we assume that the income is fully spanned and consider combinations of income-stock and income-bond correlations satisfying (15) which are admittedly far from estimated values and the general benchmark values. We assume $\rho_{Sr} = -\rho_{SB} = 0$ and consider different combinations where the income-asset correlations have the same absolute values, which must then be $1/\sqrt{2} \approx 0.7071$, and either the same sign or opposite signs, i.e. $\rho_{yr}, \rho_{yS} \in \{-0.7071, +0.7071\}$. Table 3 reports information on the magnitude of human capital and optimal strategies for three wealth-to-income ratios (all positive) and four different combinations of the two income-asset correlation coefficients.[10] We provide portfolio weights for the bond, the stock, and cash and the consumption-income ratio. The portfolio weights for the bond and the stock are decomposed into the different components. Given that both the stock-bond correlation and the risk premium on the bond are zero, there is no speculative demand for the bond. For comparison, with the parameters given, the optimal speculative position in absence of labor income will be 25% in the stock and 75% in cash. Due to the demand for hedging changes in the investment opportunity set, the optimal portfolio of the investor without labor income will be 25% in the stock, 68.6% in the 10-year bond, and 6.4% in cash.

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[10] The integrals appearing in the expression for the income multiplier and the optimal strategies are computed numerically using Romberg's method of order 10.

[Table 3 about here.]

First, note the magnitude of human capital and the portfolio weights. For a consumer-investor with a 30-year horizon and a current wealth equal to 20% of current annual income, the total human capital may be more than 60 times current financial wealth. (Of course, it will be even higher for individuals with an average income growth higher than the assumed 3%.) The human capital is heavily dependent on the correlation structure. An income stream negatively correlated with stock returns is very valuable in the same way as an asset paying counter-cyclical dividends will be highly priced. The income-bond correlation is of minor importance for the human capital. Given the large human capital the optimal investments as a fraction of current financial wealth can be extreme. In the tables we see several portfolio weights of over 100, i.e. over 10,000 percent of financial wealth. Note that the bond position is far more extreme than the stock position. For higher wealth-income ratios financial wealth constitutes a larger fraction of total wealth so the optimal portfolio weights (fractions of financial wealth) are reduced in magnitude and will asymptotically approach the optimal portfolio weights in the absence of labor income. In all cases considered in Table 3, the optimal consumption rate is higher than current income. For low current wealth the optimal consumption per year can even exceed the sum of current wealth and current income, i.e. consumption is financed in part by borrowing. This is due to the individual's desire to smooth consumption over the life-cycle. In Section 3.2, we look at the effects of imposing short-sales constraints and limits to borrowing.

Next, consider the different components of the optimal asset demands in Table 3. The magnitude of the speculative demands for stocks (and bonds, if there was any speculative demand) is only affected through the size of the human capital which varies significantly over the different combinations of correlations. The hedge term in the bond demand basically hedges total wealth against changes in investment opportunities and is therefore also affected through the size of human capital. These terms in the optimal portfolios are therefore of the same sign for all correlation pairs considered. The sign of the first correction term in the bond demand and the (only) correction term in the stock demand is determined by the sign of the contemporaneous correlation of the bond and the stock, respectively, with the income rate. The magnitudes of these terms are determined by the human capital, but also the volatilities of the assets relatively to the volatility of income. Since the bond price is less volatile than the stock price, it takes a larger bond position than stock position to "undo" the income rate risk. Therefore, the magnitude of this hedge term is larger for the bond than for the stock. Finally, the second correction term in the bond demand is due to the interest rate sensitivity of human capital. With the benchmark value $\xi_1 = 0.25 < 1$, human capital substitutes a long position in the bond so the correction term is negative as explained previously.

In Table 4, we study how the results are affected by the interest rate sensitivity of the expected income growth rate, i.e. the parameter ξ_1. We look at three values of ξ_1, namely -1, 0, and 1. In each case, the parameter ξ_0 is then fixed so that the average income growth rate $\xi_0 + \xi_1 \bar{r}$ is 3%. With $\xi_1 = 1$ and thus $\xi_0 = 1\%$, the expected income growth rate will be 2% [4%] when the real interest rate is 1% [3%] so the business cycle variations in income growth are still relatively small. The results in the table are for the case, where both the income-interest rate and the income-stock correlation are given by 0.7071, but the effects of varying (ξ_0, ξ_1) are very similar for the other correlation pairs. The decomposition of expected income growth rate affects the magnitude of human capital and thus also optimal strategies but this effect is relatively small as long as ξ_1 is modest and the initial interest rate is near the long-term average of 2%. In addition, ξ_1 is directly influencing the second correction term of the bond (and, consequently, also the demand for cash) as already indicated in the discussion above. As we can see from the table, this effect can be quite substantial and emphasizes the need to understand the behavior of labor income over the business cycle.

[Table 4 about here.]

The sensitivity of the results to the time horizon is illustrated in Table 5 again for correlations $\rho_{yr} = \rho_{yS} = 0.7071$. Of course, the human capital and therefore the magnitude of all the asset demand components are increasing in the time horizon. Note that with the assumed parameters, the optimal stock weight for a given wealth-income ratio is actually decreasing in the time horizon, but as discussed earlier this depends on the stock market premium, the risk aversion, and the correlations.

[Table 5 about here.]

We have also studied the sensitivity of results to the level of interest rates. The current interest rate has some effect on the value of human capital through the discounting of future income and this will affect all the components of the optimal portfolios. The link between human capital and the initial interest rate is affected by the value of ξ_1. In addition, the interest rate level affects the hedge of wealth against changes in investment opportunities. As long as the expected income growth is relatively little sensitive to the interest rate, the human capital and the optimal strategies are relatively stable when varying the initial interest rate near the long-term average of 2%.

4.2 Unspanned income, liquidity constraint

In this section we use the benchmark parameter values of Table 1 including the zero income-asset correlations so that the income risk is unspanned. Compared to the case with spanned income, the zero correlations eliminate the correction term in the stock demand and the first correction term in the bond demand, which will considerably reduce the magnitude of the total portfolio weight both of the stock and of the bond (and, consequently, the weight of cash). We also impose the liquidity constraint that financial wealth has to stay non-negative. The liquidity constraint will generally lower the value the individual associates with a given future income stream since it limits the use of future income for consumption smoothing. Hence the speculative asset demands and the interest rate hedge demand for the bond will be dampened relative to the unconstrained case. Of course, the liquidity constraint will have the largest effect on the optimal strategies for individuals with low financial wealth. We also consider the effects of imposing short-sales constraints in addition to the liquidity constraint. As in the previous section, the optimal strategies are only little sensitive to the current short-term interest rate near the long-term average so we only report strategies for $r = \bar{r}$.

Table 6 illustrates the optimal strategies for different values of the wealth-income ratio and different levels of the relative risk aversion coefficient. First consider the left-hand part of the table where only the liquidity constraint is imposed. The optimal portfolio weights and the consumption-income ratio are generally dampened considerably relative to the complete market case (compare for example with Table 3), in particular for low wealth-income ratios where the liquidity constraint is "nearby." For a fixed wealth-income ratio the optimal investment in the stock [bond] is decreasing [increasing] in the risk aversion consistent with traditional investment advice. Although less extreme, the optimal investment strategies still involve large positions in the bond and the stock financed by a large loan.

The results in the right-hand part of Table 6 are for the case where also short-sales constraints are imposed. For a low wealth-income ratio and low risk aversion it is optimal for the investor to invest the entire financial wealth in the stock market. Although the bond is desirable for different hedging motives, it does not enter the optimal constrained portfolio for the states and parameters considered in this table. The stock is too attractive for speculative motives. For higher risk aversion and for higher wealth-income ratios a positive bond position becomes optimal. With a high enough risk aversion, the motive for hedging total wealth against changes in investment opportunities is so strong that the bond dominates the optimal portfolio even when short-sales constraints are imposed.

[Table 6 about here.]

In Table 7 the optimal strategies for time horizons of 0.5, 2, and 10 years are illustrated for an individual with a relative risk aversion of 4 so that they are comparable to middle panel of Table 6 in which the time horizon is 30 years. Note that the optimal fraction of financial wealth invested in the stock market is increasing in the time horizon. With a horizon of six months, the short-sales constraints are not binding. For hedging purposes a short-term investor wants a relatively large position in cash, especially if the wealth-income ratio is high, since cash is closer to the truly risk-free asset than the long-term bond. For longer horizons, the 10-year zero-coupon bond is closer to the risk-free asset over the investor's horizon. Fixing the maturity of the bond, we see that, when short-sales constraints are imposed, the optimal bond weight is decreasing [increasing] in the horizon for low [high] wealth-income ratios.

[Table 7 about here.]

In Table 8 we study how the optimal constrained strategies depend on the business cycle sensitivity of the expected income growth rate. For a given ξ_1, we pick ξ_0 so that the total expected income growth is kept at $\xi_0 + \xi_1 \bar{r} = 3\%$. Just as in the complete markets case (Table 4), we see that the portfolio weight of the stock is almost independent of ξ_1 but that the relative allocation between the bond and cash varies with ξ_1 in order to obtain the desired interest rate sensitivity of total wealth. Individuals with pro-cyclical expected income growth (positive ξ_1) should invest more in bonds and less in cash than individuals with counter-cyclical expected income growth.

[Table 8 about here.]

In Table 9 we investigate how the results are affected by the correlation ρ_{yr} between shocks to labor income and shocks to the short-term interest rate. For the case with wide constraints on the portfolio weights considered in the left panel of the table, we see that the portfolio weight in the stock is roughly independent of this correlation parameter. With a positive ρ_{yr}, the contemporaneous correlation between the human capital and the bond is negative so the explicit investment in the bond should be higher. Hence, other things equal, the bond weight will be increasing in ρ_{yr}.

[Table 9 about here.]

Finally, Table 10 illustrates the effects of varying the correlation ρ_{yS} between the labor income rate and the stock returns. Again, the results are driven by the fact that the correlation parameter determines the extent to which human capital substitutes the stock or, in other words, how well the stock can hedge shocks to labor income. With a negative ρ_{yS}, the optimal portfolio weight on the stock is thus significantly higher than with a positive ρ_{yS}.

[Table 10 about here.]

4.3 Incorporating life-cycle variations in labor income

We will now consider the case where the expected income growth rate, $\xi_0(t)$, depends on age. We adopt estimation results from Cocco, Gomes, and Maenhout (2005) and Campbell and Viceira (2002) and assume that the average labor income profile over the working life (age 20 to 65 years) of an individual can be described by a third-order polynomial in age. The coefficients of the polynomial depend on the educational level, categorized as either "No High school", "High school", and "College." In the discrete-time framework of Cocco, Gomes, and Maenhout (2005), it is assumed that the retirement income level for age 66 and higher is proportional to the income level at age 65 with a replacement rate, $\bar{P}^i$, where the superscript i indicates education category. In our continuous-time setting we simply linearly interpolate the income level in the one-year period between age 65 and age 66. As discussed in more detail in Appendix A the expected income growth level can then be written as

$$\xi_0(t) = \bar{\xi}_0 + \begin{cases} b^i + 2c^i t + 3d^i t^2 & \text{if } 20 \leq t \leq 65 \\ -(1 - \bar{P}^i) & \text{if } 65 < t < 66 \\ 0 & \text{if } t \geq 66 \end{cases} \tag{31}$$

The estimated polynomial life-cycle income profiles are illustrated in Figure 2, and the polynomial coefficients used in the figure are reproduced from Cocco, Gomes and Maenhout (2002) in Table 11.

[Figure 2 about here.]

We will use $\bar{\xi}_0 = 0.02$ and the three sets of coefficients listed in Table 11 representing three different levels of education. We take the income rate volatility to be $\sigma_y(t) = 0.10$ for $t \leq 65$ and $\sigma_y(t) = 0$ for $t \geq 65$. For simplicity we continue to assume that the agent has a fixed terminal date, which we set to age 80. We take the relative risk aversion to be 2 and the time preference rate to be 0.03.

[Table 11 about here.]

The Figures 3, 4, and 5 show how the optimal investment strategy for an individual with the income profile of a high school graduate varies over the life for wealth-income ratios of 0.2, 1, and 5, respectively. All results are for an interest rate of 2%. For a given wealth-income ratio the portfolio weights in the stock and the bond decrease up to the retirement, where the future income becomes fully certain and thus replaces a bond investment. However, over the life-cycle the wealth-income ratio of an individual will also vary. Typically, the wealth-income ratio is low for young individuals, then the ratio increases since the individual wants to save for retirement, and then it tends to decrease again towards and into retirement. Of course, the precise path of optimal portfolios over the life-cycle will also depend on the realized returns on the investments and the realized income.

[Figure 3 about here.]

[Figure 4 about here.]

[Figure 5 about here.]

In Figure 6 we have depicted the optimal portfolio weights in the bond and the stock of individuals in the three different educational groups for the case of a wealth-income ratio equal to 1. After retirement, the optimal portfolio weights are the same, but before retirement we see substantial differences between the individuals. However, if we impose a strict no borrowing constraint at all times, the optimal portfolios of all three individuals will consist of 100% in stocks. As discussed earlier, this will be different for other levels of risk aversion and other values of key parameters.

[Figure 6 about here.]

5 Concluding remarks

This paper has demonstrated that the relative allocation to stocks, bonds, and cash is significantly affected by the presence of uncertain labor income. First, the human capital can substantially magnify the speculative demand for both stocks and bonds as well the intertemporal hedging demand for bonds, of course accompanied by significant borrowing. Second, the demand for stocks and bonds are corrected to the extent that the human capital implicitly substitutes stocks and bonds, respectively. In our modeling framework the correction of the stock demand is solely determined by the contemporaneous correlation between shocks to stock returns and shocks to labor income, which apparently is close to zero for typical individuals. The correction of the bond demand is not only due to a possible contemporaneous correlation between bond returns and labor income but also due to the fact that human capital is generally interest rate dependent. Our analysis emphasizes that the business cycle variations of labor income growth are crucial for the interest rate sensitivity of human capital and, consequently, for the relative allocations to long-term bonds and cash. The optimal asset allocation of an individual with a pro-cyclical labor income is substantially different form the optimal asset allocation of an individual with a counter-cyclical labor income. Good individual asset allocation advice requires identification of the magnitude and risk characteristics of the individual's labor income.

Our analysis can be extended in several interesting directions although with added computational complexity. One extension would be to allow the investor to purchase a house that can serve as collateral for a mortgage loan, thus avoiding a strict no borrowing constraint. This would basically call for a combination of our model with a set-up like that considered by Yao and Zhang (2005). Another extension is to allow for several stocks. We expect that a larger fraction of the income rate variations can be hedged using multiple risky assets, but we have no information about the typical correlations between a labor income stream and individual stocks and, hence, it is unclear how large a fraction of the risk of a typical labor income stream that can be hedged in the financial markets.[11]

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[11] Allowing for multiple stocks may "help" as indicated by a small example. Assume n stocks that are similar in the sense that they have identical expected rates of return ψ_i, identical volatilities σ_i, identical correlations ρ_{iB} with the bond, identical correlations ρ_{yi} with the labor income rate, and all pairwise stock-stock correlations are equal to ρ. Then the income process is spanned whenever

$$(\rho_{yi} - \rho_{iB}\rho_{yB})^2 = \left(1 - \rho_{yB}^2\right)\left(\frac{1-\rho}{n} + \rho - \rho_{iB}^2\right).$$

The value of ρ_{yi} for this equation to hold is decreasing in n, the number of stocks.

A Details of the calibration of the model

In line with Cocco, Gomes, and Maenhout (2005) (henceforth referred to as CGM) and Campbell and Viceira (2002), the log-income is decomposed into a personal idiosyncratic component and a common labor income component,

$$\log y_t^i = P^i(t) + u_t + v_t^i. \tag{32}$$

where the deterministic component $P^i(t)$ reflects the expected life-cycle income of investors with the same characteristics as individual i, u_t is a common stochastic income factor, and v_t^i is an individual-specific stochastic component which is assumed uncorrelated across individuals.

In order to have the model in a form consistent with the income process in (10), we assume that

$$du_t = (\xi_{u0} + \xi_{u1} r_t - \frac{1}{2}\sigma_u^2)dt + \sigma_u dz_{ut} \tag{33}$$

and

$$dv_t^i = (\xi_{v0}^i + \xi_{v1}^i r_t - \frac{1}{2}{\sigma_v^i}^2)dt + \sigma_v^i dv_{vt}^i. \tag{34}$$

Furthermore, following CGM, we will in our calibration approach assume that the deterministic personal income component $P^i(t)$ is described by a third-order polynomial in time (age),

$$P^i(t) = a^i + b^i t + c^i t^2 + d^i t^3$$

where a^i, b^i, c^i, and d^i are constant parameters.[12] The polynomial form of $P^i(t)$ only applies for age 20 until 65 (years). In the discrete-time framework of CGM it is thus assumed that the retirement income level for age 66 and higher is proportional to the income level at age 65 with a replacement rate, $\bar{P}^i$. We will adopt this form, and in our continuous-time setting we simply linearly interpolate the income level in the one-year period between age 65 and age 66. Since most of our calibrations are based on the PSID data used by CGM, we will adopt their estimated third-order polynomial structures and replacement rates for groups characterized by three different educational backgrounds: "No Highschool", "Highschool", and "College". The polynomial life-cycle income profiles are illustrated in Figure 2, and the polynomial coefficients used in the figure are reproduced from CGM in Table 11.

With the above decomposition and assumptions, it can be inferred (by an application of Ito's lemma) that the individual income process is of the form in (10) with,

$$\xi_0^i(t) = \xi_{u0} + \xi_{v0}^i + \frac{dP^i(t)}{dt} = \xi_{u0} + \xi_{v0}^i + \begin{cases} b^i + 2c^i t + 3d^i t^2 & \text{if } 20 \le t \le 65 \\ -(1-\bar{P}^i)P^i(65) & \text{if } 65 < t < 66 \\ 0 & \text{if } t \ge 66 \end{cases}$$

and the income process has constant volatility with ${\sigma_y^i}^2 = \sigma_u^2 + {\sigma_v^i}^2$. Furthermore, let l_t^i denote the stochastic part of the income process, $l_t^i = \log y_t^i - P^i(t) = u_t + v_t^i$. Then,

$$dl_t^i = (\xi_0^i + \xi_1^i r_t - \frac{1}{2}{\sigma_y^i}^2) + \sigma_y^i dz_{yt}^i \tag{35}$$

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[12] Our following estimation approach is also set up with an eye on controlling for family-specific fixed effects such as marital status and family size when relevant, as do CGM in their Table 1. For example, in our analysis of the individual household PSID income time-series this is achieved by conditioning on the first income observation, which thus implies that we do not take subsequent changes in family-specific fixed effects into account when estimating the parameters of the individual household income dynamics.

where $\xi_0^i = \xi_{u0} + \xi_{v0}^i$.

As noted earlier, we will carry out two kinds of calibrations. Some of the calibrations are based on individual time series of income data. In these we estimate the parameters of the residual income process in (35) separately for all individual households in the PSID database. These calibrations will give some insight into the dispersion of the parameter values of expected income growth, interest sensitivity, and income volatility (reflected in the parameters ξ_0, ξ_1, and σ_y) across individuals with different characteristics.

However, following ideas of, e.g., CGM we also carry out calibrations based on aggregating data across individuals that basically assume that the parameters ξ_0, ξ_1, and σ_y are the same across individuals in different educational groups. Formally, assume that the idiosyncratic component in (34) is a random walk component in the sense that $\xi_{v0}^i = \xi_{v1}^i = 0$. This implies by assumption that all individual income processes have the same drift parameters as the common stochastic component u_t. Now, by averaging across individuals in any given year, one can construct the aggregate stochastic income component $\bar{l}_t = u_t$ (where it is used that the idiosyncratic components v_t^i on average cancel out). We have,

$$d\bar{l}_t = (\xi_0 + \xi_1 r_t - \frac{1}{2}\sigma_u^2) + \sigma_u dz_{ut} \tag{36}$$

where $\xi_0 = \xi_{u0}$ and $\xi_1 = \xi_{u1}$.

The parameters of the aggregate stochastic component of labor income as well as correlations and parameters of real interest rates and stock prices are estimated by maximum likelihood. The relevant dynamics are described by the real interest rate dynamics in (1), the stock dynamics of (8), and an income process of the form in (36) (or, equivalently, (35)). Let $Y_t = (\bar{l}_t, \log S_t, r_t)'$, then the relevant dynamics can be summarized by the linear stochastic differential equation,

$$dY_t = (A + BY_t)dt + V d\hat{z}_t \tag{37}$$

where

$$A = \begin{pmatrix} \xi_0 - \frac{1}{2}\sigma_u^2 \\ \psi - \frac{1}{2}\sigma_S^2 \\ \kappa\bar{r} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & 1 \\ 0 & 0 & -\kappa \end{pmatrix}, \quad VV' = \begin{pmatrix} \sigma_u^2 & \rho_{uS}\sigma_u\sigma_S & \rho_{ur}\sigma_u\sigma_r \\ \rho_{uS}\sigma_u\sigma_S & \sigma_S^2 & \rho_{Sr}\sigma_S\sigma_r \\ \rho_{ur}\sigma_u\sigma_r & \rho_{Sr}\sigma_S\sigma_r & \sigma_r^2 \end{pmatrix},$$

and $\hat{z}_t$ is a three-dimensional standard Brownian motion. The discrete-time solution to the linear stochastic differential equation in (37) is a VAR(1)-model of the form

$$Y_{t+\Delta} = A(\Psi, \Delta) + B(\Psi, \Delta)Y_t + \tilde{\epsilon}_{t+\Delta}, \;\; \tilde{\epsilon}_{t+\Delta} \sim N(0, \Omega(\Psi, \Delta)) \tag{38}$$

where $\Psi = (\xi_0, \xi_1, \psi, \kappa, \bar{r}, \sigma_y, \sigma_S, \sigma_r, \rho_{yS}, \rho_{yr}, \rho_{Sr})$ denotes the set of parameters to be estimated. The functions $A(\Psi, \Delta)$, $B(\Psi, \Delta)$, and $\Omega(\Psi, \Delta)$, which determine the relevant moments of the VAR(1)-model, can be obtained in closed-form using the general solution formula for linear stochastic differential equations (see, e.g., Karatzas and Shreve (1988, pp. 354-357). Based on the VAR(1)-model in (38), and the expressions for the moments, the relevant likelihood function for maximization with respect to the parameters in Ψ is formulated in a straightforward manner.[13]

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[13] The numerical maximum likelihood estimation is carried out using the software program GAUSS. In the evaluation of the first and second order moments we used the analytical evaluation tools in the software program Mathematica and pasted the relevant analytical results into the GAUSS program.

In the calibration approach with aggregated income across individuals, the drift parameters for individuals are by assumption identical to those in the aggregated income process. The drift parameters can therefore by estimated from this constructed process. (Also, note that this aggregated PSID income process is conceptually very similar to a traditional macro economic income index, as for example tabulated in the NIPA tables of the National Economic Accounts.) In general, however, the volatility on aggregate income processes will understate the individual income uncertainty since the idiosyncratic noise components v_t^i are cancelled out. Following conceptual ideas of, e.g., CGM, one can define

$$\Delta l_t^i = (l_{t+\Delta}^i - l_t) - \mathrm{E}_t[(l_{t+\Delta}^i - l_t)]$$

where the expectations are calculated based on the drift parameter estimates of ξ_0 and ξ_1 obtained by maximum likelihood, as described above. We can now calculate the empirical version of $\mathrm{Var}(\Delta l_t^i)$ across all individuals and time periods and deduce the value of σ_y that this corresponds to given our assumed income process. This is our estimate of the individual income volatility. Furthermore, based on this estimate of σ_y and the estimates from the analysis of aggregate income, and assuming that the idiosyncratic income component is uncorrelated with the stock index and interest rates, one can deduce the implied correlation coefficients for individual household income: $\rho_{yS} = \rho_{uS}\frac{\sigma_u}{\sigma_y}$ and $\rho_{yr} = \rho_{ur}\frac{\sigma_u}{\sigma_y}$.

B Proof of Proposition 1

The process $\hat{z} = (\hat{z}_r, \hat{z}_S)^\top$ defined by

$$\hat{z}_t = z_t + \lambda t$$

is a standard $(n+1)$-dimensional Brownian motion under the risk-neutral probability measure. Therefore, the risk-neutral dynamics of the short-term interest rate is

$$dr_t = \left(\hat{\phi} - \kappa r_t\right) dt - \sigma_r \, d\hat{z}_{rt},$$

where $\hat{\phi} = \kappa \bar{r} + \sigma_r \lambda_1$. This implies that

$$r_u = e^{-\kappa[u-t]} r_t + \frac{\hat{\phi}}{\kappa}\left(1 - e^{-\kappa[u-t]}\right) - \int_t^u \sigma_r e^{-\kappa[u-v]} \, d\hat{z}_{rv}.$$

Applying the Fubini rule for interchanging the order of integration, we get

$$\int_t^s r_u \, du = \left(\frac{r_t}{\kappa} - \frac{\hat{\phi}}{\kappa^2}\right)\left(1 - e^{-\kappa[s-t]}\right) + \frac{\hat{\phi}}{\kappa}(s-t) - \frac{\sigma_r}{\kappa} \int_t^s \left(1 - e^{-\kappa[s-u]}\right) d\hat{z}_{ru}. \tag{39}$$

The income dynamics under the risk-neutral probability measure is

$$dy_t = y_t \left[\left(\hat{\xi}_0(t) + \xi_1 r_t\right) dt + \sigma_y(t) \left\{\rho_{yB} \, d\hat{z}_{rt} + \hat{\rho}_{yS}^\top \, d\hat{z}_{St}\right\}\right],$$

where $\hat{\xi}_0(t) = \xi_0(t) - \sigma_y(t) \rho_{yP}^\top \lambda$ and $\hat{z}_t = z_t + \lambda t$, and hence

$$\begin{aligned} y_s = y_t \exp\Bigg\{ &\int_t^s \left(\hat{\xi}_0(u) + \xi_1 r_u - \frac{1}{2}\sigma_y(u)^2\right) du \\ &+ \int_t^s \sigma_y(u) \rho_{yB} \, d\hat{z}_{ru} + \int_t^s \sigma_y(u) \hat{\rho}_{yS}^\top \, d\hat{z}_{Su} \Bigg\}. \end{aligned} \tag{40}$$

Combining (39) and (40) we get

$$\begin{aligned} y_s e^{-\int_t^s r_u \, du} = y_t \exp\Bigg\{ &\int_t^s \left(\hat{\xi}_0(u) - \frac{1}{2}\sigma_y(u)^2\right) du + (\xi_1 - 1)\left(r_t - \frac{\hat{\phi}}{\kappa}\right) b(s-t) \\ &+ \frac{\hat{\phi}}{\kappa}(\xi_1 - 1)(s-t) + \int_t^s \sigma_y(u) \hat{\rho}_{yS}^\top \, d\hat{z}_{Su} \\ &+ \int_t^s \left(\sigma_y(u) \rho_{yB} - (\xi_1 - 1)\sigma_r b(s-u)\right) d\hat{z}_{ru} \Bigg\}. \end{aligned} \tag{41}$$

The exponent on the right hand side is normally distributed and applying the standard rule for expectations of exponentials of normal random variables we get

$$\mathrm{E}_t^{\mathbb{Q}}\left[y_s e^{-\int_t^s r_u \, du}\right] = y_t e^{F(t,s) + (\xi_1 - 1) b(s-t) r_t}$$

for some easily computed function $F(t,s)$. Applying (2), we get

$$\mathrm{E}_t^{\mathbb{Q}}\left[y_s e^{-\int_t^s r_u \, du}\right] = y_t e^{F(t,s) - (\xi_1 - 1) a(s-t)} \left(B^s(r_t, t)\right)^{1-\xi_1}.$$

Defining $\ln h(t,s) = F(t,s) - (\xi_1 - 1) a(s-t)$ and integrating over s, we arrive at (17).

C Properties of the function $G(r,t)$

Lemma 1 *The function* $G(r,t)$ *defined by* (24) *has the following properties:*

(a) $G(r,t)$ *is increasing in* T*,*

(b) $G(r,t)$ *is decreasing in* r *if* $\gamma > 1$ *and increasing in* r *if* $\gamma < 1$*.*

In the proof we assume for notational simplicity that $\varepsilon = 0$ so that the individual has no utility from terminal wealth.

Proof: (a) Computing the derivative $\frac{\partial G}{\partial T}$, we can see that it will be positive whenever

$$b(T-t)\int_t^T f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds > \int_t^T b(s-t)f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds,$$

which is indeed the case since b is increasing.

(b) Computing the derivative $\frac{\partial G}{\partial r}$, we see that it will be positive if and only if

$$\begin{aligned}(\gamma-1)\Bigg[&\left(\int_t^T f(s-t)b(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right)^2\\ &-\left(\int_t^T b(s-t)^2 f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right)\left(\int_t^T f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right)\Bigg] > 0.\end{aligned}$$

The term in the square brackets is negative due to the Cauchy-Schwarz Inequality:

$$\begin{aligned}&\left(\int_t^T f(s-t)b(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right)^2\\ &\qquad=\left(\int_t^T \left[f(s-t)^{\frac{1}{2}}\left(B^s(r,t)\right)^{\frac{\gamma-1}{2\gamma}}\right]\left[b(s-t)f(s-t)^{\frac{1}{2}}\left(B^s(r,t)\right)^{\frac{\gamma-1}{2\gamma}}\right]ds\right)^2\\ &\qquad\le\left(\int_t^T f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right)\left(\int_t^T b(s-t)^2 f(s-t)\left(B^s(r,t)\right)^{\frac{\gamma-1}{\gamma}}\,ds\right).\end{aligned}$$

Consequently, $\frac{\partial G}{\partial r}$ is positive if $\gamma < 1$ and negative if $\gamma > 1$. □

D Details on the numerical solution method

We solve the highly non-linear PDE (28) in the following way. We set up an equally spaced lattice in (x, r, t)-space defined by the grid points

$$\{(x_i, r_j, t_n) \mid i = 0, 1, \ldots, I;\ j = 0, 1, \ldots, J;\ n = 0, 1, \ldots, N\},$$

where $x_i = i\Delta x$, $r_j = r_0 + j\Delta r$, and $t_n = n\Delta t$ for some fixed positive spacing parameters Δx, Δr, and Δt. Since wealth and income are restricted to non-negative values $x_0 = 0$ is a natural lower bound for $x = W/y$, while the highest value x_I is an artificial upper bound. We place the long-term interest rate level $\bar{r}$ in the middle of the grid, $r_{J/2} = \bar{r}$, and since the model allows for interest rates of all real values, we introduce an artificial lower bound, r_0, and an artificial upper bound, r_J. Since the numerical approximation is likely to be relatively imprecise near the artificial bounds, we pick the values of these bounds so that it is highly unlikely that the state moves from the starting point (x, r), located near the center of the grid, to one of the imposed bounds. We denote the approximated value function in the grid point (x_i, r_j, t_n) by $F_{i,j,n}$ and use similar notation for the controls (portfolio weights and scaled consumption) and other state-dependent functions.

We apply a backward iterative procedure starting at the terminal date T, where we first set $F_{i,j,N} = \varepsilon x_i^{1-\gamma}/(1-\gamma)$. At any earlier time t_n, we know the approximated value function at time t_{n+1}, i.e. $F_{i,j,n+1}$ for all $i = 0, 1, \ldots, I$ and all $j = 0, 1, \ldots, J$, and the optimal controls at time t_{n+1}, i.e. $\hat{c}_{i,j,n+1}$ and $\pi_{i,j,n+1}$ for all $i = 0, 1, \ldots, I$ and all $j = 0, 1, \ldots, J$. To find the approximated value function and the optimal controls at time t_n, we first make an initial guess of the optimal controls $\hat{c}_{i,j,n}$ and $\pi_{i,j,n}$ for all (i, j). Since optimal controls do not tend to vary dramatically over time, a good initial guess is $\hat{c}_{i,j,n} = \hat{c}_{i,j,n+1}$ and $\pi_{i,j,n} = \pi_{i,j,n+1}$. In the PDE (28), we can remove the sup-term if we substitute the optimal controls into the curly brackets, and then solve for the value function. Applying this idea we set up a finite difference approximation of the PDE without the sup-operator. In the finite difference approximation we use so-called "up-wind" approximations of the derivatives, which tends to stabilize the approach. For each (i, j) in the interior of the grid we thus obtain the equation

$$
\begin{aligned}
\hat{\delta}_{j,n}F_{i,j,n} = {} & \frac{\hat{c}_{i,j,n}^{1-\gamma}}{1-\gamma} + D_t^+F_{i,j,n} + D_r^+F_{i,j,n}\left(\kappa[\bar{r}-r_j]^+ - (1-\gamma)\rho_{yr}^-\sigma_{yn}\sigma_r\right) \\
& - D_r^-F_{i,j,n}\left(\kappa[\bar{r}-r_j]^- - (1-\gamma)\rho_{yr}^+\sigma_{yn}\sigma_r\right) + \frac{1}{2}\sigma_r^2 D_r^2 F_{i,j,n} \\
& + D_x^+F_{i,j,n}\Bigg\{(1-\hat{c}_{i,j,n})^+ + x_i\left[((1-\xi_1)r_j)^+ + \xi_{0n}^- + \gamma\sigma_{yn}^2\right] \\
& \qquad + x_i\sigma_B\left(\pi_{i,j,n}^B(\lambda_r - \gamma\sigma_{yn}\rho_{yB})\right)^+ + x_i\sigma_S\left(\pi_{i,j,n}^S\left(\frac{\psi}{\sigma_S} - \gamma\rho_{yS}\sigma_{yn}\right)\right)^+\Bigg\} \\
& - D_x^-F_{i,j,n}\Bigg\{(1-\hat{c}_{i,j,n})^- + x_i\left[((1-\xi_1)r_j)^- + \xi_{0n}^+\right] \\
& \qquad + x_i\sigma_B\left(\pi_{i,j,n}^B(\lambda_r - \gamma\sigma_{yn}\rho_{yB})\right)^- + x_i\sigma_S\left(\pi_{i,j,n}^S\left(\frac{\psi}{\sigma_S} - \gamma\rho_{yS}\sigma_{yn}\right)\right)^-\Bigg\} \\
& + \frac{1}{2}x_i^2 D_x^2 F_{i,j,n}\left(\pi_{i,j,n}^\top\Sigma\Sigma^\top\pi_{i,j,n} + \sigma_{yn}^2 - 2\sigma_y(t)[\pi_{i,j,n}^B\sigma_B\rho_{yB} + \pi_{i,j,n}^S\sigma_S\rho_{yS}]\right) \\
& + x_i\sigma_r D_{xr}^+F_{i,j,n}\left((\pi_{i,j,n}^B)^-\sigma_B + (\pi_{i,j,n}^S\rho_{SB})^-\sigma_S + \rho_{yr}^-\sigma_{yn}\right) \\
& - x_i\sigma_r D_{xr}^-F_{i,j,n}\left((\pi_{i,j,n}^B)^+\sigma_B + (\pi_{i,j,n}^S\rho_{SB})^+\sigma_S + \rho_{yr}^+\sigma_{yn}\right),
\end{aligned}
\tag{42}
$$

where

$$
D_t^+F_{i,j,n} = \frac{F_{i,j,n+1} - F_{i,j,n}}{\Delta t},
$$

$$
D_x^2F_{i,j,n} = \frac{F_{i+1,j,n} - 2F_{i,j,n} + F_{i-1,j,n}}{(\Delta x)^2}, \qquad D_r^2F_{i,j,n} = \frac{F_{i,j+1,n} - 2F_{i,j,n} + F_{i,j-1,n}}{(\Delta r)^2},
$$

$$
D_x^+F_{i,j,n} = \frac{F_{i+1,j,n} - F_{i,j,n}}{\Delta x}, \qquad D_x^-F_{i,j,n} = \frac{F_{i,j,n} - F_{i-1,j,n}}{\Delta x},
$$

$$
D_r^+F_{i,j,n} = \frac{F_{i,j+1,n} - F_{i,j,n}}{\Delta r}, \qquad D_r^-F_{i,j,n} = \frac{F_{i,j,n} - F_{i,j-1,n}}{\Delta r},
$$

$$
D_{xr}^+F_{i,j,n} = \frac{1}{2\Delta x\Delta r}\left(2F_{i,j,n} + F_{i+1,j+1,n} + F_{i-1,j-1,n} - F_{i+1,j,n} - F_{i-1,j,n} - F_{i,j+1,n} - F_{i,j-1,n}\right),
$$

$$
D_{xr}^-F_{i,j,n} = -\frac{1}{2\Delta x\Delta r}\left(2F_{i,j,n} + F_{i+1,j-1,n} + F_{i-1,j+1,n} - F_{i+1,j,n} - F_{i-1,j,n} - F_{i,j+1,n} - F_{i,j-1,n}\right).
$$

Adding similar equations for all points of the boundary of the grid, we get a system of $(I+1)(J+1)$ equations (one for each grid point) that we can solve for the $(I+1)(J+1)$ values $F_{i,j,n}$; more information on the solution of the equations is given below. Given that solution we compute a new guess on the optimal controls $\hat{c}_{i,j,n}$, $\pi_{i,j,n}$ from the first-order conditions from the maximization in the PDE (28), i.e. the Equations (29) and (30) in the unrestricted case, again using finite difference approximations of the derivatives based on the newly computed guess on the value function $F_{i,j,n}$. Given the new guess on the optimal controls, we solve again the system of equations and obtain a new guess on the value function at time t_n. We continue these iterations until the largest relative change in the value function over all (i,j) relative to the previous iteration is below some small threshold (we use 0.1%). Then we continue to time t_{n-1}. Typically, the solution requires 2-4 iterations at each time step.

We can write the equation system that we have to solve in the form

$$
\mathbf{M}_n\mathbf{F}_n = \mathbf{d}_n,
$$

where $\mathbf{F}_n$ is the $(I+1)(J+1)$-dimensional vector of values $F_{i,j,n}$, $\mathbf{M}_n$ is a matrix of dimension $(I+1)(J+1) \times (I+1)(J+1)$, and $\mathbf{d}_n$ is an $(I+1)(J+1)$-dimensional vector of known values. The matrix will be a band matrix so that the equation system can be solved relatively fast. The width of the band depends on the way in which order the points (i,j) are taken in the vector $\mathbf{F}_n$. The two natural candidates are

$$(0,0),(1,0),(2,0),\dots,(I,0),(0,1),(1,1),(2,1),\dots,(I,1),\dots,(0,J),(1,J),(2,J),\dots,(I,J)$$

and

$$(0,0),(0,1),(0,2),\dots,(0,J),(1,0),(1,1),(1,2),\dots(1,J),\dots,(I,0),(I,1),(I,2),\dots,(I,J).$$

The first results in a matrix with a band width of $2I+5$, whereas the band width is $2J+5$ using the second enumeration of points. In the complete market case we have observed that the value function and the optimal strategies are more sensitive to wealth and income than to the interest rate, and since we expect the same in the incomplete market case, we will use $I > J$. Therefore the second enumeration is more efficient, both in relation to the number of non-zero values to be stored in the computer and in relation to the speed with which the equation system can be solved.

The numerical results stated in Section 4 are based on an implementation of the above procedure with $I = 500$, $J = 40$, and four time steps per year. The imposed bounds are $x_I = 10$, $r_0 = -0.03$, and $r_J = 0.07$.

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	NIPA income	NIPA income	NIPA income	PSID income	PSID income	PSID income	PSID income	Benchmark
parameter	(1951-2003)	(1970-1992)	(1970-1992)	(1970-1992)	No high school	High school	College	parameters
				(Panel A)				
ξ_0	0.0181	0.0123	0.0130	-0.0035	0.0026	-0.0046	-0.0058	0.025
	(0.0043)	(0.0055)	(0.0057)	(0.0040)	(0.0055)	(0.0041)	(0.0045)	
ξ_1	-0.0434	0.1246	0.1088	0.3179	0.0322	0.3589	0.4861	0.25
	(0.1996)	(0.1059)	(0.2166)	(0.1546)	(0.2124)	(0.1584)	(0.1727)	
σ_u	0.0208	0.0235	0.0231	0.0164	0.0211	0.0163	0.0204	0.02
	(0.0010)	(0.0017)	(0.0017)	(0.0025)	(0.0031)	(0.0025)	(0.0030)	
ρ_{uS}	0.1673	0.2098	0.1997	0.3755	0.1288	0.3111	0.5677	–
	(0.0663)	(0.1027)	(0.0883)	(0.1775)	(0.2054)	(0.1899)	(0.1346)	
ρ_{ur}	0.2683	0.2343	0.1806	0.1466	0.2242	0.1064	0.0810	–
	(0.0727)	(0.1009)	(0.0753)	(0.1535)	(0.1580)	(0.1591)	(0.1388)	
ψ	0.0665	0.0497	*0.0665*	*0.0665*	*0.0665*	*0.0665*	*0.0665*	0.04
	(0.0226)	(0.0373)						
κ	0.6407	0.7139	*0.6407*	*0.6407*	*0.6407*	*0.6407*	*0.6407*	0.50
	(0.1942)	(0.2703)						
$\bar{r}$	0.0155	0.0142	*0.0155*	*0.0155*	*0.0155*	*0.0155*	*0.0155*	0.02
	(0.0047)	(0.0089)						
σ_S	0.1613	0.1790	*0.1613*	*0.1613*	*0.1613*	*0.1613*	*0.1613*	0.20
	(0.0079)	(0.0132)						
σ_r	0.0218	0.0302	*0.0218*	*0.0218*	*0.0218*	*0.0218*	*0.0218*	0.02
	(0.0012)	(0.0024)						
ρ_{Sr}	0.0052	-0.0297	*0.0052*	*0.0052*	*0.0052*	*0.0052*	*0.0052*	0
	(0.0658)	(0.1175)						
				(Panel B)				
σ_y	–	–	–	0.3431	0.3612	0.3303	0.3296	0.20
ρ_{yS}	–	–	–	0.0179	0.0075	0.0154	0.0351	0
ρ_{yr}	–	–	–	0.0070	0.0131	0.0053	0.0050	0

Table 1: **Parameter estimates based on aggregated income data.**
Standard errors in parentheses.

parameter	PSID income No high school			PSID income High school			PSID income College		
	90% fractile	Median	10% fractile	90% fractile	Median	10% fractile	90% fractile	Median	10% fractile
ξ_0	0.1744	0.0390	-0.0324	0.1829	0.0271	-0.0602	0.1833	0.0277	-0.0709
		(0.0831)			(0.0797)			(0.0796)	
ξ_1	3.6789	-0.2429	-5.8429	3.7704	-0.0933	-6.2123	5.4803	0.4912	-4.2618
		(3.3421)			(2.9230)			(2.8484)	
σ_y	0.5281	0.2857	0.1268	0.5037	0.2442	0.1148	0.4960	0.2443	0.1025
		(0.0500)			(0.0435)			(0.0429)	
ρ_{yS}	0.3554	0.0073	-0.3824	0.3715	0.0286	-0.3534	0.3658	0.0440	-0.3208
		(0.2127)			(0.2169)			(0.2133)	
ρ_{yr}	0.2869	0.0097	-0.2431	0.2868	0.0152	-0.2821	0.3418	0.0148	-0.2734
		(0.1659)			(0.1675)			(0.1660)	
No. individuals		486			1273			543	
No. observations		8530			22198			9769	

Table 2: **Parameter estimates based on individual household income data.** The table reports the 10% fractile, median, and 90% fractile of different estimates for 2302 individual household income parameters obtained separately by maximum likelihood. In the estimations (based on (38)) the stock and interest rate parameters were fixed at the same values as in Table 1, i.e.: $\psi = 0.0665$, $\kappa = 0.6407$, $\bar{r} = 0.0155$, $\sigma_S = 0.1613$, $\sigma_r = 0.0218$, and $\rho_{Sr} = 0.0052$. The median standard errors are in parentheses.

	bond investment				stock investment				
W/y	corr1	corr2	hedge	total	spec	corr	total	cash	c/y
$\rho_{yr} = 0.7071$, $\rho_{yS} = 0.7071$, $M = 22.08$									
0.2	393	-76.11	76.63	393.5	27.85	-78.07	-50.22	-342.3	1.036
1	78.6	-15.22	15.88	79.26	5.771	-15.61	-9.844	-68.41	1.083
5	15.72	-3.045	3.725	16.40	1.354	-3.123	-1.769	-13.63	1.270
$\rho_{yr} = 0.7071$, $\rho_{yS} = -0.7071$, $M = 52.97$									
0.2	942.7	-191.8	182.8	933.7	66.46	187.3	253.7	-1186	2.494
1	188.5	-38.37	37.12	187.3	13.49	37.45	50.94	-237.2	2.531
5	37.71	-7.673	7.973	38.01	2.898	7.49	10.39	-47.40	2.719
$\rho_{yr} = -0.7071$, $\rho_{yS} = -0.7071$, $M = 60.62$									
0.2	-1079	-220.7	208.7	-1091	76.03	214.3	290.4	801.6	2.916
1	-215.8	-44.13	42.29	-217.6	15.41	42.87	58.27	160.4	2.954
5	-43.16	-8.827	9.007	-42.98	3.281	8.574	11.85	32.12	3.146
$\rho_{yr} = -0.7071$, $\rho_{yS} = 0.7071$, $M = 24.42$									
0.2	-434.5	-84.86	84.65	-434.7	30.77	-86.32	-55.55	491.3	1.155
1	-86.91	-16.97	17.48	-86.40	6.354	-17.26	-10.91	98.31	1.192
5	-17.38	-3.394	4.046	-16.73	1.471	-3.453	-1.982	19.71	1.380

Table 3: **The optimal strategies with a 30-year horizon for different income-asset correlations.** The table shows how the components of optimal portfolios and the consumption-income ratio depend on the decomposition of the expected income growth rate. Benchmark parameter values are used except for the income-asset correlations. The investor has a 30-year horizon, a risk aversion of 4, and a time preference rate of 3%. The expected annual income growth rate is fixed at 3%.

W/y	bond investment				stock investment			cash	c/y
	corr1	corr2	hedge	total	spec	corr	total		
$\xi_0 = 0.05, \xi_1 = -1$, $M = 20.97$									
0.2	373.3	-191.9	72.81	254.2	26.47	-74.15	-47.68	-205.5	0.9931
1	74.65	-38.38	15.11	51.39	5.493	-14.83	-9.337	-41.05	1.031
5	14.93	-7.676	3.573	10.83	1.299	-2.966	-1.667	-8.160	1.218
$\xi_0 = 0.03, \xi_1 = 0$, $M = 21.81$									
0.2	388.1	-100.1	75.68	363.7	27.51	-77.1	-49.59	-313.1	1.032
1	77.62	-20.02	15.69	73.29	5.702	-15.42	-9.718	-62.57	1.070
5	15.52	-4.004	3.687	15.21	1.34	-3.084	-1.744	-12.46	1.257
$\xi_0 = 0.01, \xi_1 = 1$, $M = 23.09$									
0.2	411.0	0	80.09	491.1	29.11	-81.64	-52.53	-437.5	1.092
1	82.19	0	16.57	98.76	6.023	-16.33	-10.31	-87.46	1.130
5	16.44	0	3.864	20.30	1.405	-3.266	-1.861	-17.44	1.318

Table 4: **The sensitivity of the optimal strategies to the decomposition of the expected income growth rate.** The table shows how the components of optimal portfolios and the consumption-income ratio depend on the decomposition of the expected income growth rate. The investor has a 30-year horizon, a risk aversion of 4, and a time preference rate of 3%. The income-asset correlations are $\rho_{yr} = \rho_{yS} = \sqrt{2}/2$.

	bond investment				stock investment				
W/y	corr1	corr2	hedge	total	spec	corr	total	cash	c/y
$T = 0.5$, $M = 0.4977$									
0.2	8.858	-0.2161	0.4890	9.13	0.8721	-1.76	-0.8875	-7.243	0.4702
1	1.772	-0.04323	0.2099	1.938	0.3744	-0.3519	0.02251	-0.9607	1.009
5	0.3543	-0.008646	0.1541	0.4998	0.2749	-0.07038	0.2045	0.2957	3.705
$T = 2$, $M = 1.962$									
0.2	34.91	-2.71	3.694	35.9	2.702	-6.936	-4.234	-30.67	0.7461
1	6.983	-0.5419	1.012	7.453	0.7404	-1.387	-0.6467	-5.807	1.022
5	1.397	-0.1084	0.4758	1.764	0.3481	-0.2774	0.07065	-0.8347	2.403
$T = 10$, $M = 9.024$									
0.2	160.6	-26.85	27.93	161.7	11.53	-31.91	-20.37	-140.3	0.9632
1	32.12	-5.37	6.071	32.82	2.506	-6.381	-3.875	-27.95	1.047
5	6.424	-1.074	1.699	7.049	0.7012	-1.276	-0.575	-5.474	1.464

Table 5: **The sensitivity of the optimal strategies to the time horizon.** The table shows how the components of optimal portfolios and the consumption-income ratio depend on the time horizon of the individual. The investor has a risk aversion of 4 and a time preference rate of 3%. The income-asset correlations are $\rho_{yr} = \rho_{yS} = \sqrt{2}/2$.

	Liquidity constraint				No short sales or borrowing			
W/y	π_B	π_S	π_0	c/y	π_B	π_S	π_0	c/y
	$\gamma = 2$							
0.2	-2.386	9.724	-6.337	0.7791	0	1	0	0.7519
1	-0.7166	3.804	-2.087	0.8906	0	1	0	0.8408
5	0.2039	1.099	-0.3028	1.327	0	1	0	1.296
	$\gamma = 4$							
0.2	2.213	7.044	-8.256	0.6441	0	1	0	0.6343
1	0.9809	1.812	-1.793	0.7261	0	1	0	0.7217
5	0.6721	0.4707	-0.1428	1.154	0.5353	0.4647	0	1.154
	$\gamma = 6$							
0.2	5.023	4.444	-8.467	0.5388	0	1	0	0.5464
1	2.029	1.188	-2.218	0.6126	0.1135	0.8865	0	0.6395
5	4.879	0.4356	-4.315	0.9055	0.7350	0.2481	0.01693	1.143

Table 6: **Optimal strategies for different levels of risk aversion.** The table shows the optimal strategies for investors with a relative risk aversion equal to 2, 4, and 6, both for the case with only the liquidity constraint and for the case where the short sales constraint is also imposed. The investor has a time preference rate of 0.03 and a 30-year horizon. Benchmark parameter values are used. The current short-term interest rate is 2%.

	Liquidity constraint				No short sales or borrowing			
W/y	π_B	π_S	π_0	c/y	π_B	π_S	π_0	c/y
$T = 0.5$, $M = 0.5013$								
0.2	0.1954	0.6815	0.123	0.4386	0.1954	0.6815	0.1230	0.4386
1	0.1574	0.3595	0.4831	1.014	0.1574	0.3595	0.4831	1.014
5	0.1495	0.2762	0.5743	3.687	0.1495	0.2762	0.5743	3.687
$T = 2$, $M = 2.020$								
0.2	0.8529	2.166	-2.019	0.6998	0	1	0	0.699
1	0.4501	0.6755	-0.1255	1.006	0.3294	0.6706	0	1.006
5	0.3701	0.3511	0.2789	2.387	0.3701	0.3511	0.2789	2.387
$T = 10$, $M = 10.53$								
0.2	1.578	5.987	-6.565	0.7575	0	1	0	0.7475
1	0.8009	1.619	-1.42	0.8696	0	1	0	0.866
5	0.6209	0.5247	-0.1455	1.378	0.4812	0.5188	0	1.378

Table 7: **Optimal strategies for different time horizons.** The table shows the optimal strategies for investors with various time horizons both for the case with only the liquidity constraint and for the case where the short sales constraint is also imposed. The investor has a time preference rate of 0.03 and a relative risk aversion of 4. Benchmark parameter values are used. The current short-term interest rate is 2%.

	Liquidity constraint				No short sales or borrowing			
W/y	π_B	π_S	π_0	c/y	π_B	π_S	π_0	c/y
$\xi_0 = 0.05$, $\xi_1 = -1$, $M = 36.48$								
0.2	-20.91	6.439	15.47	0.6533	0	1	0	0.6384
1	-4.878	1.796	4.082	0.7397	0	1	0	0.7304
5	0.3762	0.5516	0.0721	1.144	0.0003	0.4429	0.5568	1.193
$\xi_0 = 0.03$, $\xi_1 = 0$, $M = 35.35$								
0.2	-2.923	7.045	-3.122	0.6446	0	1	0	0.6349
1	-0.2167	1.82	-0.603	0.7265	0	1	0	0.7224
5	0.5141	0.4805	0.0054	1.15	0.5142	0.4805	0.0054	1.15
$\xi_0 = 0.01$, $\xi_1 = 1$, $M = 34.99$								
0.2	16.75	6.709	-22.46	0.6446	0	1	0	0.6327
1	4.644	1.83	-5.474	0.7275	0	1	0	0.7223
5	1.764	0.5278	-1.291	1.115	0.5814	0.4186	0	1.183

Table 8: **The sensitivity of the optimal strategies to the decomposition of the expected income growth rate.** The table shows the optimal strategies for different decompositions of the expected income growth rate both for the case with only the liquidity constraint and for the case where the short sales constraint is also imposed. The investor has a time preference rate of 0.03, a relative risk aversion of 4, and a 30-year horizon. Benchmark parameter values are used. The current short-term interest rate is 2%.

	Liquidity constraint				No short sales or borrowing			
W/y	π_B	π_S	π_0	c/y	π_B	π_S	π_0	c/y
$\rho_{yr} = -0.25$, $M = 35.93$								
0.2	-27.88	6.178	22.71	0.6630	0	1	0	0.6416
1	-7.143	1.859	6.284	0.7493	0	1	0	0.7308
5	-0.5169	0.484	1.033	1.174	0	0.4725	0.5275	1.168
$\rho_{yr} = 0.25$, $M = 34.47$								
0.2	23.16	5.761	-27.92	0.6520	0	1	0	0.6273
1	8.897	1.825	-9.722	0.7406	0	1	0	0.7132
5	1.856	0.4825	-1.339	1.166	0.5791	0.4209	0	1.144

Table 9: **The sensitivity of the optimal strategies to the correlation between income and interest rate.** The table shows the optimal strategies for two non-benchmark values of the correlation coefficient between the labor income and the short-term interest rate both for the case with only the liquidity constraint and for the case where the short sales constraint is also imposed. The investor has a time preference rate of 0.03, a relative risk aversion of 4, and a 30-year horizon. The current short-term interest rate is 2%.

	Liquidity constraint				No short sales or borrowing			
W/y	π_B	π_S	π_0	c/y	π_B	π_S	π_0	c/y
$\rho_{yS} = -0.25$, $M = 41.36$								
0.2	1.694	9.702	-10.4	0.6865	0	1	0	0.6506
1	0.9136	3.277	-3.19	0.7882	0	1	0	0.7525
5	0.6540	0.7445	-0.3986	1.230	0.2902	0.7098	0	1.220
$\rho_{yS} = 0.25$, $M = 30.17$								
0.2	2.493	0.25	-1.743	0.6219	0.8162	0.1838	0	0.6218
1	1.011	0.25	-0.2609	0.6992	0.7599	0.2401	0	0.6992
5	0.6819	0.25	0.06815	1.116	0.6819	0.25	0.06814	1.116

Table 10: **The sensitivity of the optimal strategies to the correlation between income and stock.** The table shows the optimal strategies for two non-benchmark values of the correlation coefficient between the labor income and the stock index both for the case with only the liquidity constraint and for the case where the short sales constraint is also imposed. The investor has a time preference rate of 0.03, a relative risk aversion of 42, and a 30-year horizon. The current short-term interest rate is 2%.

	No High school	High school	College
a^i	-2.1361	-2.1700	-4.3148
b^i	0.1684	0.1682	0.3194
c^i	-0.00353	-0.00323	-0.00577
d^i	0.000023	0.000020	0.000033
$\bar{P}^i$	0.88983	0.68212	0.93887
Initial age 20 level (in 2002 US dollar)	17,763	19,107	13,912

Table 11: **Coefficients in the life-cycle income polynomials.**

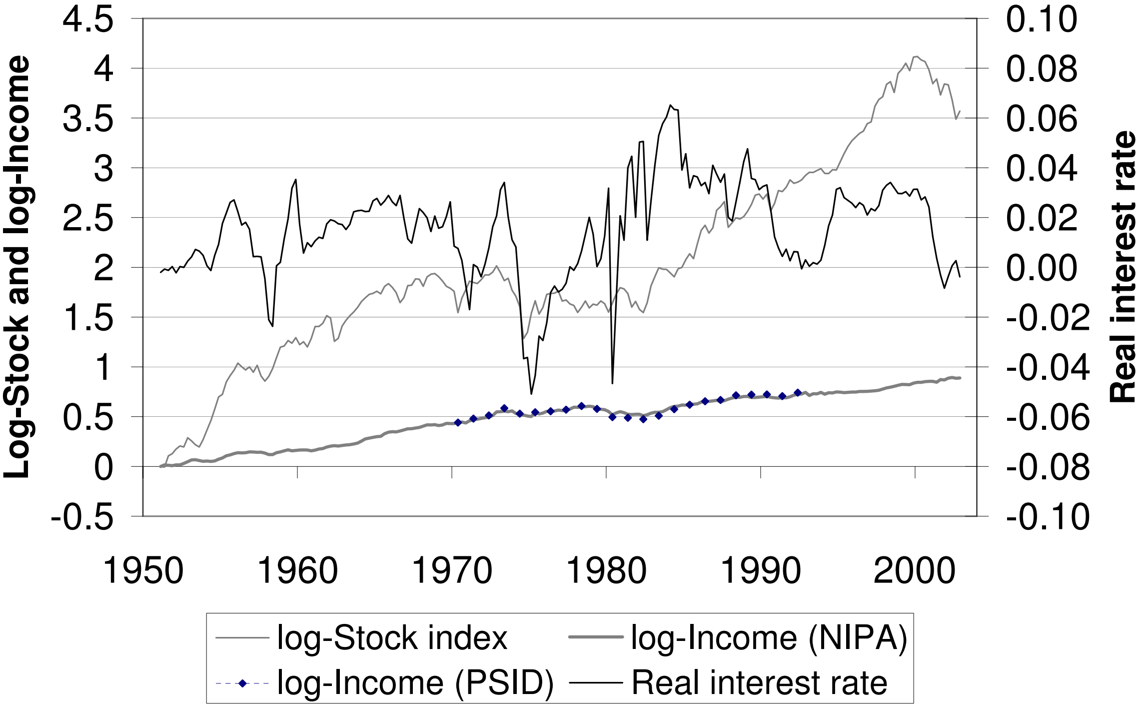

Figure 1: **Data series on income, stock index and real interest rates.** Further description...

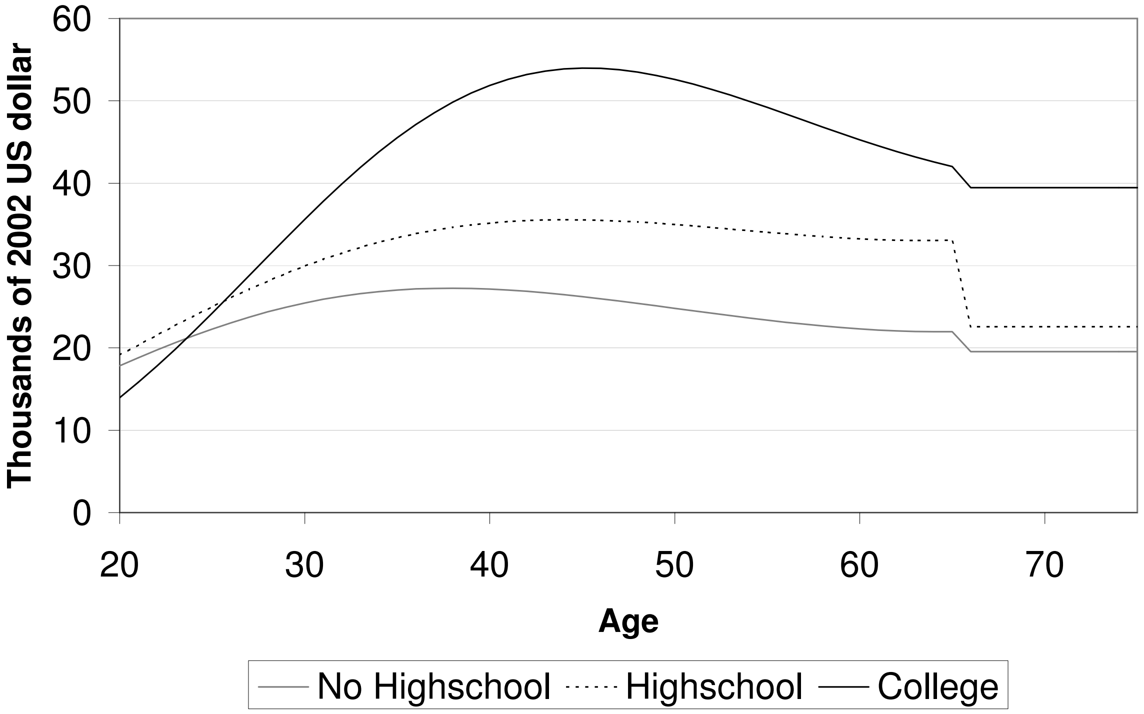

Figure 2: **Calibrated life-cycle income polynomials.**

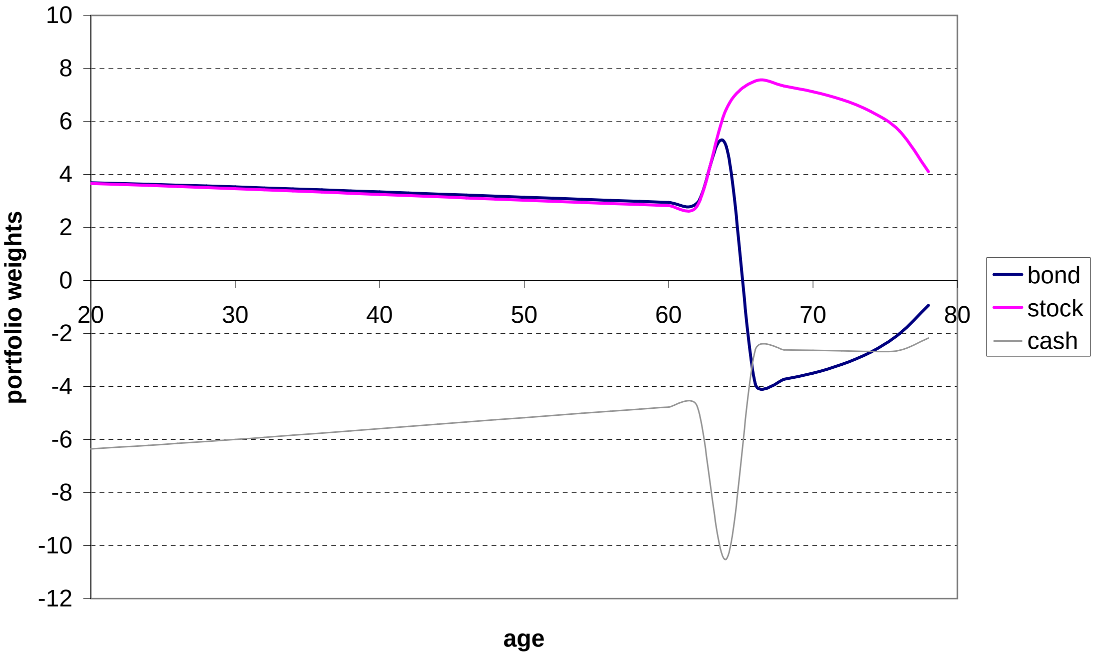

Figure 3: **The optimal investment strategies over the life of an individual with high school education for a wealth-income ratio of 0.2.**

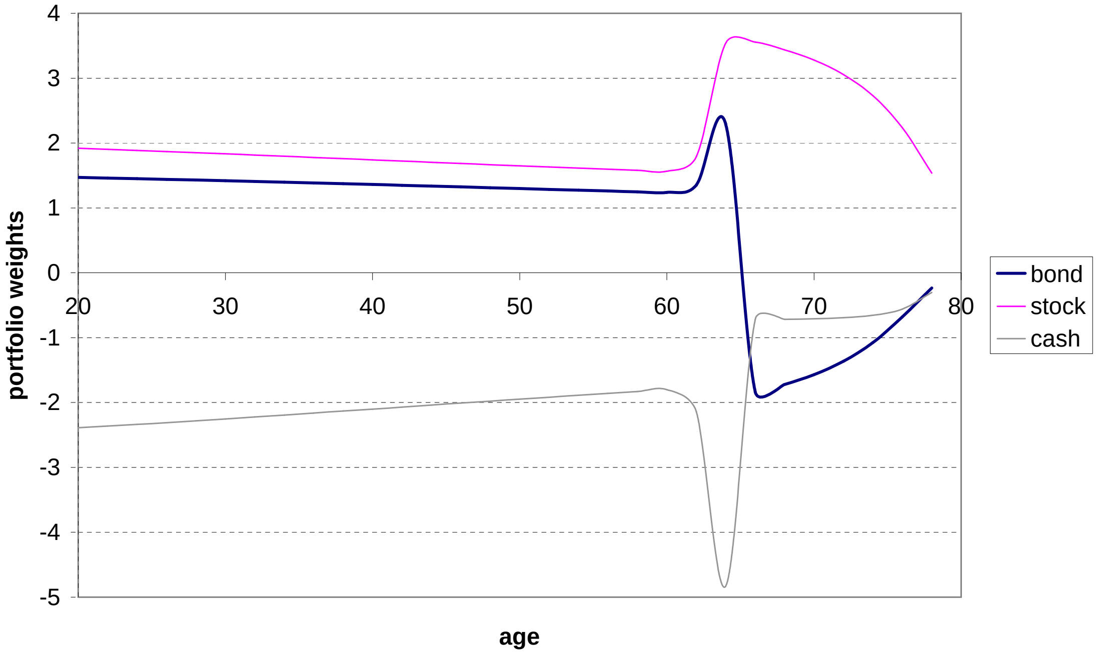

Figure 4: **The optimal investment strategies over the life of an individual with high school education for a wealth-income ratio of 1.**

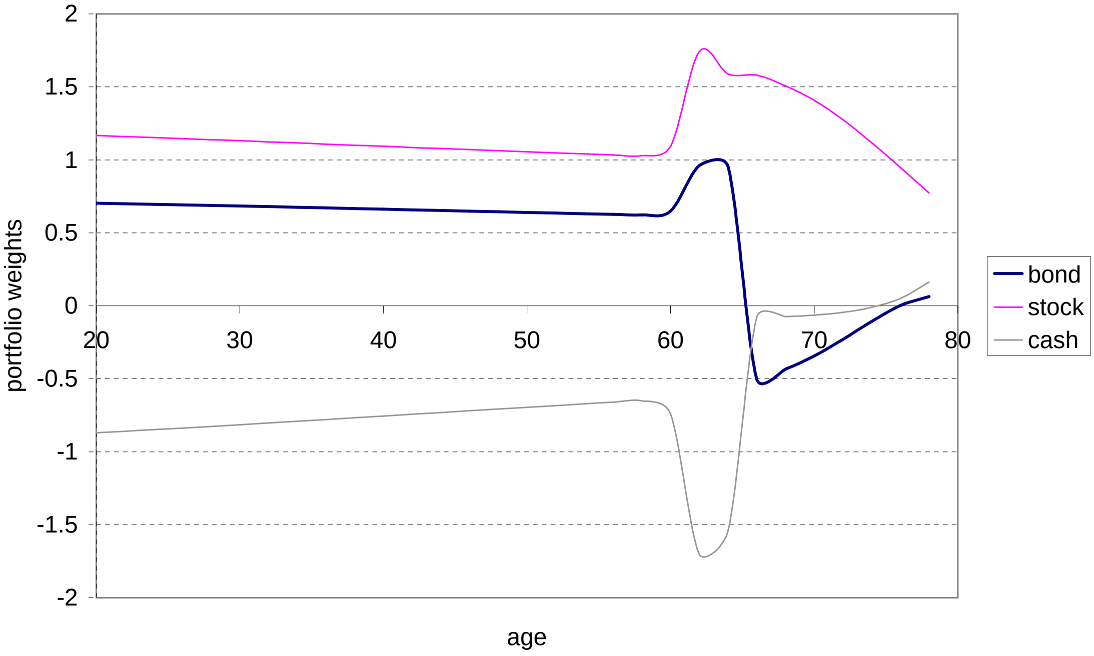

Figure 5: **The optimal investment strategies over the life of an individual with high school education for a wealth-income ratio of 5.**

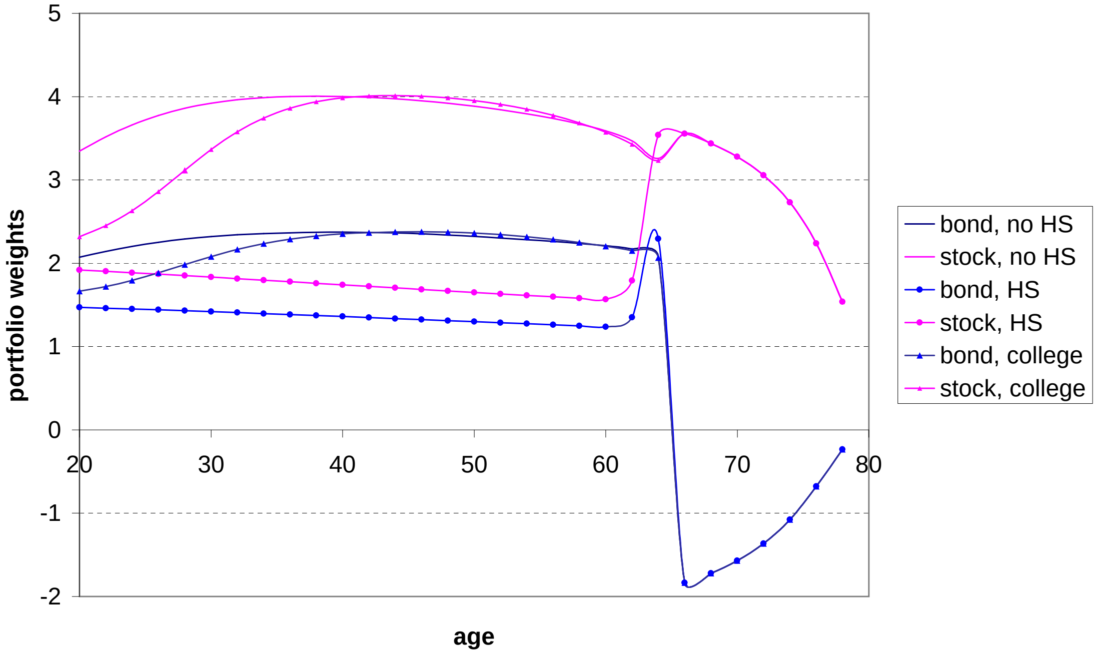

Figure 6: **The optimal investment strategies over the life of an individual for the three educational groups, all for a wealth-income ratio of 1.**