

Traffic Light Options

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Abstract

This paper introduces, prices, and analyzes *traffic light options*. The traffic light option is an innovative structured OTC derivative developed independently by several London-based investment banks to suit the needs of Danish life and pension (L&P) companies, which must comply with the *traffic light solvency stress test system* introduced by the Danish Financial Supervisory Authority (DFSA) in June 2001. This monitoring system requires L&P companies to submit regular reports documenting the sensitivity of the companies' base capital to certain pre-defined market shocks – the *red* and *yellow light* scenarios. These stress scenarios entail drops in interest rates as well as in stock prices, and traffic light options are thus designed to pay off and preserve sufficient capital when interest rates and stock prices fall *simultaneously*. Sweden's FSA implemented a traffic light system in January 2006, and supervisory authorities in other European countries are contemplating similar regulation. Traffic light options are therefore likely to attract the attention of a wider audience of pension fund managers in the future. Focusing on the valuation of the traffic light option we set up a Black-Scholes-Vasicek model to describe stock market and interest rate dynamics, and analyze the traffic light option in this framework.

1 Introduction

Pension funds and life insurance companies throughout the world have faced enormous challenges in recent years. During much of the 1990s interest rates were falling and thus turning promised benefits and guaranteed return features embedded in many life and pension (L&P) contracts into huge liabilities for the issuing companies.

As if to make matters worse winds of change blew through the offices of supervisory authorities and regulators who introduced tougher reporting demands and decided to monitor the business more closely. In the new millennium financial reporting standards have undergone a gradual reformation from allowing transactions based (historical cost) accounting to requiring that companies report assets and liability values at fair market values (marking-to-market).¹ The new fair value based accounting standards have removed one of the sector's last possibilities to conceal temporary solvency problems by applying actuarial smoothing techniques to balance sheet entries, and have put the spotlight on the problems of the L&P business: According to recent estimates European life insurers currently face a combined shortfall of about 100bn EUR when measured against new fair value based Solvency II capital requirements.² The funding deficit in corporate America's pension funds is estimated at a staggering 350bn USD.³

In Denmark L&P companies felt the strain of a massive amount of issued 4.5% annual after-tax return guarantees become more severe during the late 1990s as interest rates continued to fall. In the beginning of the new millennium Danish L&P companies finally initiated hedge strategies involving the purchase of protection against further interest rate drops in the form of interest rate derivatives such as receiver swaps, receiver swaptions, and CMS floors. The reported market value of Danish L&P companies' holdings of (mainly interest rate related) financial derivatives increased from 0 in early 2000 to DKK 81 billion (about USD 13 billion) in mid-2005.⁴ But L&P companies also responded to the threat of insolvency induced by a prolonged low-interest rate scenario by increasing their equity exposure. While some politicians, academics, and commentators of the financial press expressed their concern

¹For further discussion of this issue see e.g. Jørgensen (2004).

²See Mercer Oliver Wyman (2004) and The Economist (2004a).

³See The Economist (2004b) and Watson-Wyatt (2003).

⁴Source: Danmarks Nationalbank, www.nationalbanken.dk.

over such a strategy and over the increased asset-liability mismatch that it would imply, some managers of L&P companies publicly complained that “capturing the equity premium” was the only way in which the promised pension benefits could eventually be honored.

The Danish Financial Supervisory Authorities (DFSA) apparently felt it could not let this latent “asset substitution” problem develop and responded in June 2001 by introducing a new risk based solvency reporting system, which quickly became known as the *traffic light system*. The traffic light system is a scenario-based supervision tool which requires Danish L&P companies to submit semi-annual reports on the effect on their base capital of adverse changes in key market variables as defined in the “red” and “yellow light” scenarios. The red light scenario basically involves a 70bps decrease in the interest rate level, a 12% decline in stock prices, and an 8% decrease in real estate investment values. If an L&P company’s base capital falls below a given critical level in this scenario, then the company is “flashed” with a “red light”.⁵ In practice this implies strict monitoring by the DFSA, and the company will be required to submit more frequent (monthly) solvency reports. The yellow light scenario is more severe. It involves a 100bps decrease in interest rates, a 30% decline in general stock prices, and a 12% decrease in real estate investment values. If the base capital drops below the critical level in this scenario, the company receives yellow light status and will be required to submit quarterly solvency reports. Companies which can withstand the yellow light scenario without experiencing solvency problems will operate in “green light”.⁶

The introduction of the traffic light system in mid-2001 marked the beginning of a period with a sharply increased focus on proper asset-liability management in Danish L&P companies. And for those that did not adjust their equity exposure in accordance with the new rules, an embarrassing lesson was learned when equity markets collapsed following the “9/11” terrorist attacks in New York and Washington. Many pension funds – including the one with Danish finance and economics professors as members (!) – found themselves having to report red light status at the end of 2001.⁷

⁵The critical level is approximately equal to 4.5% of the pension obligations (the technical provision).

⁶Inspired by the DFSAs rather positive experience with this system, Sweden’s FSA has decided to implement a similar traffic light system for Swedish L&P companies as of January 1, 2006, see e.g. Menon (2005).

⁷When the traffic light system was introduced in mid-2001 about 30% of all Danish L&P companies had either red or yellow light status. Three years later in June 2004 all companies operated under green light, see e.g. DFSA (2005) and www.ftnet.dk. The most recent report from DFSA (mid-2005) reveals that 10 companies are in yellow light. None have red light status.

As the traffic light system became understood and implemented, investment officers of pension funds and (particularly!) their contacts in investment banks' derivative offices began thinking about strategies and instruments that could help satisfy the pension fund managers' appetite for equity risk while at the same time observing the interest rate risk and the traffic light stress tests. One outcome of this process has been the invention of a new class of derivative instruments sometimes referred to as *correlation products*. The fundamental idea behind these instruments has been to construct derivatives which pay off in the traffic light scenarios but in such a way that over-hedging is avoided. Over-hedging may result if the L&P company buys protection against downside interest rate and stock market risk separately. A consequence of this could be a payoff (from an interest rate option, for example) when it is not really needed (because of an offsetting capital gain on the equity portfolio). The challenge is thus to structure products which pay off more when interest rates and stock prices fall simultaneously, and less, if anything, when only one of the variables moves adversely.⁸ It is intuitively clear that the correlation across interest rate and equity markets is of vital importance when designing and pricing such products; hence their name.

In this paper we analyze a particularly interesting subclass of the class of correlation derivatives which we label *traffic light options*. In their purest form traffic light options are European-style derivatives with a payoff function which is the product of a standard equity put option and an interest rate *floorlet*. These options have been offered to Danish L&P companies by London-based investment banks such as Dresdner Kleinwort Wasserstein and Goldman Sachs.

The remainder of the paper is organized as follows. In the next section we provide further details on traffic light options and we develop the dynamic framework in which we will analyze these options. A closed formula for the traffic light option is derived using change of numeraire techniques. To the best of our knowledge this formula is new to the option pricing literature. Section 3 discusses the implementation of our pricing formula and provides numerical examples and illustrations. Section 4 further analyzes the usefulness of traffic light options as a hedging instrument for the typical L&P company's balance sheet. Finally, Section 5 concludes.

⁸Hedging against shocks to real estate values is usually ignored, both because it is impractical and because real estate investments constitute an insignificant part of total portfolios.

2 Model development

In this section we introduce and formalize the basic traffic light option design, and we set up a dynamic modelling framework within which this type of option can be priced using standard assumptions of perfect markets and absence of arbitrage.

Let $r(t)$ and $S(t)$ denote the time t short interest rate and the equity portfolio value respectively. As briefly discussed in the introduction, the *traffic light option* has been designed to provide protection against simultaneous declines in the interest rate and in the equity values. More specifically, the traffic light option is a European-style derivative issued at time 0 and with time T payoff given as

$$V(r, S, T) = [\bar{r} - r(T)]^+ \cdot [\bar{S} - S(T)]^+, \quad (1)$$

where \bar{r} and \bar{S} are the constant *strike* levels for the interest rate and equity portfolio value, respectively. The payoff function is thus the product of the payoffs of a standard interest rate *floorlet* and a standard equity put option. An obvious consequence of this payoff specification is that a non-zero payoff occurs if and only if both the interest rate and the equity portfolio value are below their respective strike levels at the maturity date. Figure 1 illustrates the payoff function of the traffic light option.⁹

⁹A piecewise linear – and in many respects easier manageable – payoff function could be obtained by specifying $V(r, S, T) = [\bar{r} - r(T)]^+ 1_{\{S(T) < \bar{S}\}} + [\bar{S} - S(T)]^+ 1_{\{r(T) < \bar{r}\}}$. This payoff function has the serious disadvantage, however, of being discontinuous around the strike levels. To the best of the author's knowledge it is not seen in practice either, and is therefore not analyzed further in this paper.

Figure 1

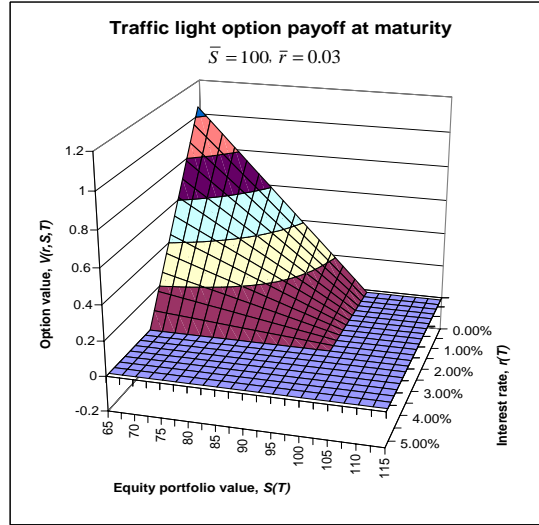


Figure 1: Payoff function of traffic light option

2.1 A dynamic model framework

In order to price and analyze the traffic light option we must formulate a model of interest rate uncertainty as well as of equity market risk. One such widely used framework – see e.g. Briys and de Varenne (1997) (valuation of pension and life insurance liabilities), Longstaff and Schwartz (1995), Shimko, Tejima, and van Deventer (1993) (valuation of risky debt), Sørensen (1999) (dynamic asset allocation), and Wilmott (1998) (valuation of convertible bonds) – is the following continuous time formulation:

$$dr(t) = \kappa(\theta - r(t)) dt + \sigma_r d\mathcal{W}_r(t) \quad (2)$$

$$dS(t) = \mu S(t) dt + \sigma_S S(t) d\mathcal{W}_S(t), \quad (3)$$

with

$$d\mathcal{W}_r(t) \cdot d\mathcal{W}_S(t) = \rho dt. \quad (4)$$

In this dynamic system $\mathcal{W}_r(t)$ and $\mathcal{W}_S(t)$ are standard correlated Wiener processes supported by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ on the finite time interval $[0, T]$. \mathcal{P} is the physical probability measure.

From (2) it is seen that the short interest rate is assumed to follow a mean-reverting Gaussian Ornstein-Uhlenbeck (or Vasicek (1977)) process with constant parameters κ , θ , and

σ_r . The parameter κ is the *force of mean-reversion*, θ is the *level of mean-reversion*, and σ_r is the interest rate volatility.

The equity portfolio value evolves as a geometric Brownian motion (as in Black and Scholes (1973)) with constant expected rate of return μ , and constant volatility parameter σ_S .¹⁰ Finally, ρ in (4) denotes the constant coefficient of correlation between the interest rate and equity value processes. Intuitively, this is a key parameter in modelling the option type in question here.

2.2 Valuation of contingent claims

If we assume perfect market conditions (continuous trading, absence of transaction costs, no short selling constraints etc.) and let $C(r, S, t)$ denote the time t value of a general derivative which depends only on time and on the contemporaneous values of the two state variables, then an application of Ito's lemma combined with the usual no arbitrage requirement (see e.g. Hull (2006) or Wilmott (1998)) establishes that $C(r, S, t)$ must satisfy the following partial differential equation:

$$\begin{aligned} [\kappa(\theta - r) - \sigma_r \lambda_r(\cdot)] \frac{\partial C}{\partial r} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 C}{\partial r^2} + rS \frac{\partial C}{\partial S} \\ + \frac{1}{2} \sigma_S^2 S^2 \frac{\partial^2 C}{\partial S^2} + \sigma_S \sigma_r \rho S \frac{\partial^2 C}{\partial r \partial S} + \frac{\partial C}{\partial t} - rC = 0. \end{aligned} \quad (5)$$

In (5) $\lambda_r(\cdot)$ is the (possibly state-dependent) *market price of interest rate risk* which must be estimated empirically or of which further assumptions must be made. Following most of the literature cited previously we assume it to be constant, and we henceforth denote it simply by λ_r .¹¹

By imposing boundary conditions appropriate for a specific derivative contract one may be able to directly identify a solution for $C(\cdot)$ which satisfies equation (5) (see e.g. Shimko, Tejima, and van Deventer (1993)). Alternatively, (5) may be used as the point of departure for a (finite difference) numerical scheme (see e.g. Chapter 48 of Wilmott (1998)). However, when valuing European-style derivatives in particular, it is often more fruitful to pursue a

¹⁰For notational simplicity we do not model dividends from the equity portfolio, even though the inclusion of a constant dividend rate would be straightforward.

¹¹In the derivations underlying (5), λ_r is defined on a form where it should normally be specified as *negative*. See also Appendix A.

risk-neutral valuation approach (Harrison and Kreps (1979)). According to this approach the arbitrage free valuation functions that solve (5) have alternative probabilistic representations as expectations under the risk neutral probability measure. More specifically, with $C(r, S, T)$ being the known payoff function of a European-style derivative, the time t value of this derivative (and thus the solution to (5)) can be expressed as

$$C(r, S, t) = E^Q \left\{ e^{-\int_t^T r(s) ds} \cdot C(r, S, T) | \mathcal{F}_t \right\}, \quad t \in [0, T], \quad (6)$$

where Q denotes the equivalent risk neutral (martingale) measure and where the expectation should be formed with respect to the transformed risk neutral dynamics (see Appendix A):¹²

$$dr(t) = \kappa(\bar{\theta} - r(t)) dt + \sigma_r dW_r^Q(t) \quad (7)$$

$$dS(t) = r(t)S(t) dt + \sigma_S S(t) dW_S^Q(t), \quad (8)$$

with

$$dW_r^Q(t) \cdot dW_S^Q(t) = \rho dt, \quad (9)$$

and where $\bar{\theta} \equiv \theta - \frac{\sigma_r \lambda_r}{\kappa}$.

A quick example which will also become useful shortly, is the class of default free zero-coupon (pure discount) bonds which pay out \$1 at the maturity date $\bar{T} \in [0, T]$. These securities can be considered as somewhat degenerate “derivatives” since their time t price, $P(r, t; \bar{T})$, will depend only on time and the interest rate, and we may thus write

$$P(r, t; \bar{T}) = E^Q \left\{ e^{-\int_t^{\bar{T}} r(s) ds} \cdot 1 | \mathcal{F}_t \right\}. \quad (10)$$

Given (7), the right-hand side of (10) can easily be calculated (see Vasicek (1977)) to establish a closed formula for the zero-coupon bond price:

$$P(r, t; \bar{T}) = \exp \left\{ -(\bar{T} - t) \left(\bar{\theta} - \frac{\sigma_r^2}{2\kappa^2} \right) + \Psi(\bar{T} - t) \left(\bar{\theta} - \frac{\sigma_r^2}{2\kappa^2} - r(t) \right) - \Psi^2(\bar{T} - t) \frac{\sigma_r^2}{4\kappa} \right\}, \quad (11)$$

where $\Psi(x) = \frac{1 - e^{-\kappa x}}{\kappa}$.

¹²According to the Equivalent Martingale Measure theorem (see e.g. Hull (2006)), no arbitrage prevails iff there is an equivalent measure such that all asset prices are martingales when denominated in terms of the associated numeraire asset. The risk neutral approach is associated with the *money market account* $MM(t) \equiv e^{\int_0^t r(s) ds}$ as the numeraire asset. Observe that (6) is equivalent to $\frac{C(r, S, t)}{MM(t)} = E_t^Q \left\{ \frac{C(r, S, T)}{MM(T)} \right\}$ — the martingale result.

2.3 Valuing the traffic light option

As regards valuation of the traffic light option at time $t \in [0, T]$, the risk neutral technique prescribes that we calculate

$$V(r, S, t) = E^Q \left\{ e^{-\int_t^T r(s) ds} \cdot [\bar{r} - r(T)]^+ \cdot [\bar{S} - S(T)]^+ | \mathcal{F}_t \right\}. \quad (12)$$

While the right-hand side lends itself to easy numerical evaluation via Monte Carlo simulation (Boyle (1977)), the analytic computation of the expectation requires integration over the joint trivariate Q -density of $r(T)$, $S(T)$, and $\int_t^T r(s) ds$. This is clearly a formidable task.

Instead we will demonstrate the usefulness of another change of numeraire (and associated probability measure). Letting the zero-coupon bond expiring at time T (the maturity date of the option) serve as numeraire, the associated martingale measure, Q^T , is referred to as the *forward neutral measure*. Using this measure the price of the traffic light option can be represented as

$$V(r, S, t) = P(r, t; T) \cdot E^{Q^T} \left\{ [\bar{r} - r(T)]^+ \cdot [\bar{S} - S(T)]^+ | \mathcal{F}_t \right\}, \quad (13)$$

with $P(r, t; T)$ given as in (11). The associated system of factor variable dynamics can be shown (see Appendix A) to take the following form under Q^T :

$$dr(t) = \left(\kappa(\bar{\theta} - r(t)) - \sigma_r^2 \Psi(T - t) \right) dt + \sigma_r d\mathcal{W}_r^{Q^T}(t) \quad (14)$$

$$\frac{dS(t)}{S(t)} = \left(r(t) - \rho \sigma_S \sigma_r \Psi(T - t) \right) dt + \sigma_S d\mathcal{W}_S^{Q^T}(t), \quad (15)$$

with

$$d\mathcal{W}_r^{Q^T}(t) \cdot d\mathcal{W}_S^{Q^T}(t) = \rho dt, \quad (16)$$

and with $\bar{\theta}$ and $\Psi(x)$ given as before.

The change of numeraire has had two main effects. First, the discount term has been brought outside the expectation operator in the form of the price of the zero-coupon bond which is known and given in (11). Second, the dimensionality of the problem of calculating the expectation has been reduced by one. That is, to calculate the expectation we must integrate over the joint bivariate density of $r(T)$ and $S(T)$ only. As shown in the paper's Appendices B and C this is feasible. The analytical valuation formula obtained for the traffic light option is stated in the proposition below.

Proposition 1:

The time t value of the traffic light option is

$$\begin{aligned}
C(r, S, t) = & P(r, t; T) \cdot \left[(\bar{r} - m_r) \bar{S} \cdot M \left(\frac{\bar{r} - m_r}{v_r}, \frac{\ln \bar{S} - m_S}{v_S}; \rho_{rS} \right) \right. \\
& + (m_r + \sigma_{rS} - \bar{r}) e^{m_S + \frac{1}{2} v_S^2} \cdot M \left(\frac{\bar{r} - m_r}{v_r} - \rho_{rS} v_S, \frac{\ln \bar{S} - m_S}{v_S} - v_S; \rho_{rS} \right) \\
& \left. + v_r \left(e^{m_S + \frac{1}{2} v_S^2} I(v_S) - \bar{S} I(0) \right) \right], \tag{17}
\end{aligned}$$

where

$$m_r \equiv E_t^{Q^T} \{r(T)\} = r(t) + \kappa(\bar{\theta} - r(t))\Psi(T-t) - \frac{\sigma_r^2}{2}\Psi^2(T-t) \tag{18}$$

$$v_r^2 \equiv \text{Var}_t^{Q^T} \{r(T)\} = \frac{\sigma_r^2}{2\kappa} \left(1 - e^{-2\kappa(T-t)}\right) \tag{19}$$

$$\begin{aligned}
m_S \equiv E_t^{Q^T} \{\ln S(T)\} = & \ln S(t) + \left(\bar{\theta} - \frac{\sigma_r^2}{\kappa^2} - \frac{\rho\sigma_S\sigma_r}{\kappa} - \frac{1}{2}\sigma_S^2 \right) (T-t) \\
& - \left(\bar{\theta} - \frac{\sigma_r^2}{\kappa^2} - \frac{\rho\sigma_S\sigma_r}{\kappa} - r(t) \right) \Psi(T-t) + \frac{\sigma_r^2}{2\kappa} \Psi^2(T-t) \tag{20}
\end{aligned}$$

$$\begin{aligned}
v_S^2 \equiv \text{Var}_t^{Q^T} \{\ln S(T)\} = & \left(\frac{\sigma_r^2}{\kappa^2} + \sigma_S^2 + \frac{2\rho\sigma_r\sigma_S}{\kappa} \right) (T-t) \\
& - \left(\frac{\sigma_r^2}{\kappa^2} + \frac{2\rho\sigma_r\sigma_S}{\kappa} \right) \Psi(T-t) - \frac{\sigma_r^2}{2\kappa} \Psi^2(T-t) \tag{21}
\end{aligned}$$

$$\sigma_{rS} \equiv \text{Cov}_t^{Q^T} \{r(T), \ln S(T)\} = \rho\sigma_S\sigma_r\Psi(T-t) + \frac{\sigma_r^2}{2}\Psi^2(T-t) \tag{22}$$

$$\rho_{rS} \equiv \frac{\sigma_{rS}}{v_r v_S}$$

$$\begin{aligned}
I(\alpha) \equiv & E \left\{ \tilde{\varepsilon} N \left(\frac{\frac{\ln \bar{S} - m_S}{v_S} - \alpha - \rho_{rS} \tilde{\varepsilon}}{\sqrt{1 - \rho_{rS}^2}} \right) \mathbf{1}_{\tilde{\varepsilon} < \frac{\bar{r} - m_r}{v_r} - \alpha \rho_{rS}} \right\} \\
= & \int_{-\infty}^{\frac{\bar{r} - m_r}{v_r} - \alpha \rho_{rS}} u \cdot N \left(\frac{\frac{\ln \bar{S} - m_S}{v_S} - \alpha - \rho_{rS} u}{\sqrt{1 - \rho_{rS}^2}} \right) n(u) du, \tag{23}
\end{aligned}$$

and where $P(r, t; T)$ is the zero-coupon bond price given in (11), $M(\cdot, \cdot; \rho)$ is the cumulative probability in the standardized bivariate normal distribution with correlation coefficient ρ , $N(\cdot)$ is the cumulative probability in the standardized univariate normal distribution, $\tilde{\varepsilon}$ is a standard normal variate, and $n(\cdot)$ is the standard normal density function.

□

Proof of Proposition 1:

See the Appendices. Appendix B considers the joint conditional Q^T -distribution of $r(T)$ and $S(T)$, and establishes the parameter functions (means, variances, and covariance) given in (18)–(22). Appendix C contains the most basic steps in calculating the forward risk neutral expectation of the traffic light option's payoff given on the right-hand side of (13).

□

Although evaluation of the expression for the traffic light option value requires a bit of computer programming, relation (17) is a closed formula containing functions which can be accurately approximated very quickly.

The bivariate normal probability entering twice in (17) can be evaluated by numerical integration, or by one of the many highly accurate approximation formulas available.¹³ In the present paper we have relied on the specific version of Drezner's approximation function (see Drezner (1978)) given in Haug (1997). This five-point Gaussian quadrature produces values of $M(\cdot, \cdot; \rho)$ within six decimal places accuracy. A four-point Gaussian quadrature is provided in the more well-known derivatives text by Hull (2003).¹⁴

Evaluation of (17) also requires the calculation of the $I(\cdot)$ -function defined in (23). This function is a well-defined univariate integral which can be accurately approximated very quickly by standard numerical integration.

¹³See Agca and Chance (2003) for a comparison of the speed and accuracy of five different bivariate normal probability approximation schemes in an option pricing context.

¹⁴This approximation has been removed from the more recent Hull (2006), and readers are now referred instead to the author's website for a *technical note*.

3 Numerical illustrations

As indicated above, formula (17) of Proposition 1 is easily evaluated. For the various numerical illustrations below we implemented the formula by writing a tiny bit of computer code in Visual Basic for Excel, using Microsoft Excel's built-in standard functions wherever possible (such as the one for the standard normal distribution function). Although Visual Basic is notoriously slow compared to standard alternatives such as C++ and Delphi Pascal, our program executed and delivered prices virtually instantly.¹⁵

Figure 2 shows a plot of traffic light option values with 1 year to maturity for different current values of the state variables r and S . Strike levels are $\bar{r} = 0.03$ and $\bar{S} = 100$ precisely as in the earlier plot of the payoff function in Figure 1. The remaining parameters have been set to reflect a realistic situation with a current term structure of zero-coupon interest rates sloping upwards from 3% (r_0) and converging towards 4.48% (r_∞) as time goes to infinity.¹⁶ The volatility of the short interest rate, σ_r , is 2%, the mean reversion level, θ , is 4%, and the speed of mean reversion, κ , is 0.25. The market price of interest rate risk, λ_r , is set at -0.1 , and the volatility of the equity portfolio, σ_S , is 20%. The correlation between the two factor processes, ρ , is assumed to be -0.5 .

The plot in Figure 2 shows that the traffic light option value is a decreasing function in each of the state variables, r and S . Note that in contrast to what would have been the result of evaluating (12) by Monte Carlo simulation, the option value surface appears nice and smooth. This has obvious advantages in relation to, for example, the determination of numerical derivatives for use in the hedging process. The figure shows that the option values converge to zero as the state variables move further out-of-the-money, and are largest when both state variables lie below their respective strike levels. These properties of the option value function are completely as expected for a multiplicative put option such as the traffic

¹⁵The evaluation of the $I(\cdot)$ -functions of (17) by numerical integration is the more time consuming part of the computations. We approximated the integral defined in (23) by standard mid-point numerical integration using an equally spaced sequence of abscissas from a lower cut-off point of -5 up to the given upper limit. Option values reported in the tables below were obtained for a partition of the integration range into 500 sub-intervals of equal length. We computed option values in less than a second for a number of sub-intervals, N , as high as 50,000. However, convergence of the computed option values on the sixth decimal was generally obtained for much smaller N .

¹⁶From (11), $r_\infty \equiv \lim_{T \rightarrow \infty} \frac{-\ln P(r,t;T)}{T-t} = \bar{\theta} - \frac{\sigma_r^2}{2\kappa^2}$.

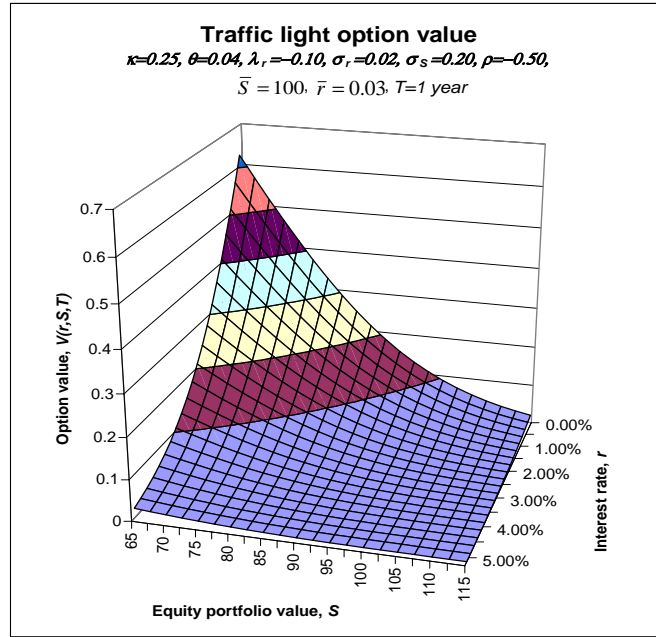


Figure 2: Traffic light option values for different current values of the state variables, r and S .

light option.

For the same set of interest rate process parameters and equity value process parameters as in Figure 2, Table 1 illustrates the dependence of at-the-money (ATM) traffic light option values on the correlation parameter, ρ , and time to maturity, T .¹⁷ The table confirms that ρ is indeed a key parameter. When correlation is strongly negative, the options are seen to be almost worthless, and option values increase sharply as ρ increases. These effects are of course closely related to the fact that the table considers ATM options for which the state variables must move in the same direction and drop below their respective strike levels in order for a payoff to obtain.

¹⁷Note that option values in Tables 1 and 2 are multiplied by 100 due to their small absolute values.

Table 1:

		Traffic Light Option values $\times 100$ Dependence on correlation and time to maturity							
		$\kappa = 0.25, \theta = 0.04, \sigma_r = 0.02, \lambda_r = -0.1, \sigma_S = 0.20$ $\bar{r} = r_0 = 0.03, \bar{S} = S(0) = \bar{S} = 100$							
		T (years)							
		0.25	0.50	1	2	3	5	10	20
ρ	-0.99	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.007
	-0.95	0.007	0.009	0.010	0.009	0.008	0.008	0.014	0.020
	-0.75	0.125	0.208	0.315	0.417	0.453	0.456	0.398	0.268
	-0.50	0.397	0.687	1.109	1.601	1.840	1.973	1.734	1.091
	-0.25	0.770	1.357	2.242	3.338	3.917	4.305	3.832	2.400
	0.00	1.232	2.190	3.660	5.527	6.544	7.257	6.475	4.047
	0.25	1.780	3.178	5.342	8.120	9.643	10.708	9.500	5.905
	0.50	2.414	4.324	7.290	11.100	13.177	14.578	12.789	7.874
	0.75	3.147	5.646	9.530	14.492	17.149	18.829	16.252	9.881
	0.95	3.827	6.873	11.596	17.570	20.693	22.504	19.102	11.477
0.99	3.980	7.148	12.054	18.240	21.451	23.271	19.678	11.793	

In the horizontal dimension Table 1 shows that (for $\rho > -0.75$) option values peak for a time to maturity between 3 and 10 years. Moreover, option values converge towards 0 as $T \rightarrow \infty$ independently of the current value of the state variables. These properties are not surprising in light of the fact that both the straight European interest rate floorlet, and the straight European equity put – when considered separately – are known to possess similar properties given the assumptions about market dynamics applied here.

Table 2 provides representative values of in-the-money (ITM), at-the-money, as well as out-of-the-money (OTM) traffic light options for a more varied span of parameter values. The option values behave as expected: They increase with the correlation, and they increase in both partial measures of volatility, σ_S and σ_r . Of course, ITM options are more valuable than ATM options, which are in turn more valuable than OTM options.

Table 2:

Traffic Light Option values $\times 100$											
$\kappa = 0.25, \theta = 0.04, \lambda_r = -0.10$											
$\bar{r} = 0.03, \bar{S} = 100, T = 1$											
			$\sigma_r = 0.01$			$\sigma_r = 0.02$			$\sigma_r = 0.03$		
			r_0			r_0			r_0		
ρ	σ_S	S_0	0.025	0.030	0.035	0.025	0.030	0.035	0.025	0.030	0.035
-0.50	0.10	95	0.925	0.390	0.139	1.696	1.080	0.658	2.602	1.897	1.356
		100	0.379	0.148	0.048	0.695	0.422	0.245	1.077	0.759	0.523
		105	0.131	0.047	0.014	0.240	0.139	0.077	0.376	0.255	0.169
	0.20	95	1.526	0.656	0.238	2.697	1.750	1.089	3.984	2.962	2.161
		100	0.990	0.410	0.143	1.743	1.109	0.676	2.577	1.889	1.358
		105	0.624	0.249	0.083	1.093	0.682	0.407	1.617	1.169	0.828
	0.30	95	2.195	0.955	0.351	3.836	2.514	1.581	5.595	4.199	3.093
		100	1.661	0.706	0.253	2.892	1.872	1.162	4.218	3.138	2.291
		105	1.245	0.517	0.180	2.162	1.382	0.847	3.152	2.325	1.682
			$\sigma_r = 0.01$			$\sigma_r = 0.02$			$\sigma_r = 0.03$		
			r_0			r_0			r_0		
ρ	σ_S	S_0	0.025	0.030	0.035	0.025	0.030	0.035	0.025	0.030	0.035
0.00	0.10	95	2.146	1.153	0.549	4.064	2.932	2.054	6.229	4.955	3.892
		100	1.170	0.622	0.294	2.252	1.610	1.117	3.512	2.768	2.154
		105	0.567	0.298	0.139	1.112	0.786	0.540	1.768	1.380	1.063
	0.20	95	3.629	1.982	0.960	6.685	4.900	3.490	9.971	8.056	6.427
		100	2.706	1.474	0.712	5.004	3.660	2.600	7.495	6.041	4.808
		105	1.974	1.073	0.517	3.667	2.675	1.896	5.515	4.435	3.521
	0.30	95	5.126	2.817	1.372	9.348	6.894	4.938	13.804	11.218	9.002
		100	4.246	2.331	1.134	7.758	5.715	4.090	11.476	9.317	7.468
		105	3.495	1.916	0.932	6.396	4.707	3.365	9.479	7.688	6.156
			$\sigma_r = 0.01$			$\sigma_r = 0.02$			$\sigma_r = 0.03$		
			r_0			r_0			r_0		
ρ	σ_S	S_0	0.025	0.030	0.035	0.025	0.030	0.035	0.025	0.030	0.035
0.50	0.10	95	3.639	2.177	1.165	6.947	5.303	3.943	10.574	8.750	7.160
		100	2.274	1.389	0.762	4.444	3.411	2.554	6.891	5.708	4.678
		105	1.272	0.794	0.447	2.560	1.975	1.489	4.069	3.370	2.764
	0.20	95	6.230	3.782	2.049	11.652	9.017	6.793	17.389	14.578	12.082
		100	4.974	3.059	1.681	9.372	7.290	5.524	14.061	11.820	9.825
		105	3.887	2.422	1.351	7.383	5.773	4.400	11.143	9.392	7.829
	0.30	95	8.705	5.294	2.868	16.142	12.534	9.471	23.901	20.115	16.731
		100	7.549	4.633	2.536	14.056	10.957	8.315	20.865	17.601	14.676
		105	6.500	4.026	2.227	12.153	9.512	7.250	18.090	15.294	12.784

4 The traffic light option as a hedging instrument

In this section we illustrate the usefulness of the traffic light option as a hedging instrument for the balance sheet of a typical pension fund or life insurance company. We start out from a representative balance sheet with a simple and typical *unmatched* (or *mismatched*) distribution of assets and liabilities. We focus on the *solvency percentage* (or *solvency ratio*) of this balance sheet – defined as free equity divided by the market value of liabilities – and study the sensitivity of this quantity to changes in the state variables. The sensitivity analysis is carried out for both a simple, unhedged balance sheet, and for a balance sheet that has been hedged with a number of appropriately designed traffic light options.

The illustration starts out at time 0 from a normalized balance sheet as illustrated in the figure below. The asset side is comprised of 20 (units of account) invested in a well-diversified stock portfolio and 80 units invested in bonds. Apart from the fact that pension funds typically have a minor part (2-8%) of their funds invested in real estate, this is a realistic asset allocation.¹⁸

Table 3

Assets		Liabilities	
Stocks	20.00	92.00	Pension obligations ($D = 20$ years)
Bonds ($D = 6$ years)	80.00	8.00	Equity (Solvency ratio: 8.70%)
Total	100.00	100.00	Total

The bond investment is further assumed to be characterized by a duration of 6 years. Again this is inspired by actual pension fund portfolio compositions. In practice pension funds often face a limited market supply of investment grade bonds with durations higher than 6-8 years, and “long” bond portfolios can therefore be difficult to acquire. On the liability side the market value of the “bond like” pension obligations is set to 92 units of account and an equal percentage of the balance sheet sum. These are very long obligations with an assumed

¹⁸Before the DFSA introduction of the traffic light stress test system, the maximum allowable equity portfolio weight in stocks was a fixed percentage. It was 40% up to 1997, and following extensive pressure on regulators from pension fund managers eager to capture the equity premium, it was raised to 50% between 1997 and 2000, cf. the discussion in the introduction.

duration of 20 years. This is not unrealistic. Actual pension fund liability durations typically vary between 15 and 25 years depending on the exact age distribution of the policy holders. In the present example the initial free equity is thus 8 units of account, and the solvency ratio thus equals $\frac{8}{92}$ or 8.70%.

Now, in order to perform the desired sensitivity analysis we assume that the market values and dynamics of the balance sheet entries, i.e. the stock portfolio, the bond investment, and the liabilities, are in accordance with the model given in (2)–(4). In particular we assume that the interest rate parameters are given as $r_0 = 0.04$, $\kappa = 0.25$, $\theta = 0.04$, $\sigma_r = 0.02$, and $\lambda_r = -0.1$. For simplicity the bond investment and the liability entries are assumed to be zero coupon bonds with maturities (=durations) equal to 6 and 20 years respectively. The present values of 80 and 92 imply initial bond and pension liability “notionals” of 103.52 and 222.52, respectively.

We can now study changes in the entries of this unhedged or “naked” balance sheet – and thus also in the solvency ratio – as a consequence of changes in the state variables, the interest rate and the value of the stock portfolio. For a first illustration suppose that the interest rate drops from 4% to 3%, and that the stock portfolio loses 30% of its value right after time 0.¹⁹ As a consequence of these changes in the state variables the value of the bond portfolio will increase to 82.53, but liabilities increase to 95.73. The equity portfolio value drops to 14, and the resulting market value based balance sheet now looks as follows.

Table 4

Assets		Liabilities	
Stocks	14.00	95.73	Pension obligations ($D = 20$ years)
Bonds ($D = 6$ years)	82.53	0.80	Equity (Solvency ratio: 0.83%)
Total	96.53	96.53	Total

As can be seen from Table 4 the solvency percentage has taken a critical blow and dropped to a mere 0.83%. A more complete picture of the solvency ratio’s response to changes in the

¹⁹The size of the shocks to the state variables is inspired by the DFSA’s yellow light scenario. Note, however, that *stricto sensu* our dynamic model is not consistent with discontinuous jumps in the state variables. Moreover, in the Vasicek model term structure movements are not parallel and long rates will always change less than the short rate.

state variables can be seen in Figure 3 below.

Figure 3

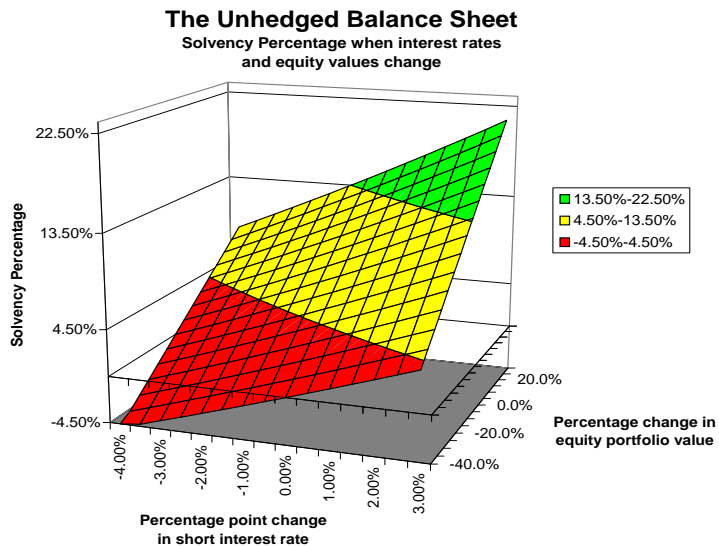


Figure 3: Sensitivity of solvency ratio to changes in the state variables in the unhedged balance sheet. $r_0 = 0.04$, $\kappa = 0.25$, $\theta = 0.04$, $\sigma_r = 0.02$, and $\lambda_r = -0.1$.

The plot in Figure 3 clearly shows how the unhedged pension fund is exposed to the risk of interest rate and stock price shocks. Without protection the solvency ratio will decrease in an almost linear fashion when the interest rate or equity values fall, and most dramatically so when both state variables fall.

Now, the traffic light option was of course created with the purpose of providing a hedge instrument for precisely these simultaneous adverse changes in the stock prices and in the interest rate. Let us therefore illustrate how – with appropriate traffic light options added – the solvency percentage will respond less dramatically to changes in S_0 and in r_0 . For this purpose we must specify numerical values for the stock portfolio volatility, σ_S , and the correlation coefficient, ρ , in order to be able to value the traffic light option. Suppose therefore that $\sigma_S = 0.20$ and that $\rho = -0.50$. Finally, suppose that 150 traffic light options with $\bar{r} = 0.04$, $\bar{S} = 21$, and $T = 6$ are acquired. Our model prices these options at 0.00715 a piece. For simplicity the total value of 1.07 for the 150 options will be financed by selling

bonds. While the liability side of the balance sheet is unaffected by this disposition, the asset side of the new balance sheet will contain a new entry and will take the following form:

Table 5

Assets		Liabilities	
Stocks	20.00	92.00	Pension obligations ($D = 20$ years)
Bonds ($D = 6$ years)	78.93		
TL options	1.07	8.00	Equity (Solvency ratio: 8.70%)
Total	100.00	100.00	Total

Of course, the initial solvency ratio is unaffected by the purchase of options and is again 8.70%.

We will now study the response of the solvency ratio of this hedged balance sheet to changes in the state variables. Consider first the (yellow light) scenario where r_0 drops from 4% to 3%, and where stocks lose 30% of their value. Following this scenario the balance sheet will look as follows:

Table 6

Assets		Liabilities	
Stocks	14.00	95.73	Pension obligations ($D = 20$ years)
Bonds ($D = 6$ years)	81.42		
TL options	4.18	3.87	Equity (Solvency ratio: 4.05%)
Total	99.60	99.60	Total

Note that solvency is again hurt, but not nearly to the same degree as before due to a significant gain on the traffic light option position. The key solvency ratio remains above the (typical) FSA requirement of 4%. In other words, with the particular design of traffic light options applied here, the pension fund will maintain a “green light” following the toughest of the stress test scenarios defined by the (Danish) FSA. A more general picture of the effect on the solvency ratio of changes in the state variables in this hedged balance sheet is illustrated in

Figure 4 below. As is clear from the plot, the effect of adding the traffic light option has been to “twist” the solvency percentage surface upwards in the area of critically low state variables. There are now no states with solvency ratios below 3%. The price for this protection is of course paid by sacrificing part of the “upside”, in the sense that in the presence of traffic light option coverage/hedging solvency will increase less in response to favorable changes in the state variables in comparison with the unhedged balance sheet situation.

Figure 4

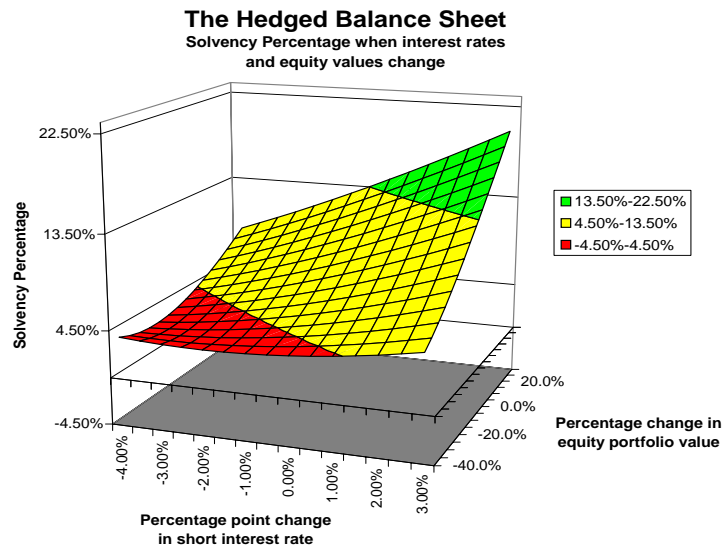


Figure 4: Sensitivity of solvency ratio to changes in the state variables in the hedged balance sheet. $r_0 = 0.04$, $\kappa = 0.25$, $\theta = 0.04$, $\sigma_r = 0.02$, $\lambda_r = -0.1$, $\sigma_S = 0.20$, and $\rho = -0.50$.

5 Conclusions

This paper has introduced, priced, and analyzed a new instrument in the landscape of exotic financial derivatives – the *traffic light option*. Traffic light options have been recently developed and introduced by a number of investment banks in order to suit the needs of Danish L&P companies, which must comply with the Danish FSA’s *traffic light stress tests* introduced in mid-2001. This scenario-based risk supervision system basically tests whether L&P companies’ base capital can withstand certain pre-defined shocks to interest rates and

to stock prices – the *traffic light scenarios*. L&P company solvency suffers when interest rates fall and when stock prices fall, and they are in double jeopardy when such falls occur simultaneously as in the DFSA's *red* and *yellow light* scenarios. Traffic light options are thus designed to pay off and provide solvency protection for L&P companies in scenarios where both interest rates and stock prices fall.

In this paper we have introduced a continuous time dynamic framework in which the traffic light option can be analyzed using standard assumptions of perfect markets and absence of arbitrage. The main contribution of this paper was a closed form solution for the value of the traffic light option. The implementation of the formula was discussed and numerical examples were provided to illustrate the usefulness of the traffic light option for hedging the typical L&P company balance sheet.

There is no doubt that traffic light options hold great potential as hedging instruments not only for Danish L&P companies but on a world-wide scale since regulatory authorities in more and more countries implement market value based financial reporting standards and solvency tests. However, one possible obstacle for the success of traffic light options concerns the estimation of the model's correlation coefficient. As the paper's numerical section has shown, the correlation between equity and interest rate risk is an enormously important determinant for the value of traffic light options. It is a parameter which is difficult to estimate accurately, and it may in fact not be constant even over short time intervals. One might therefore worry that the uncertainty surrounding the true value of this parameter may lead sellers of traffic light options (investment banks) to set premiums on the "safe side". Traffic light options may thus in turn appear "expensive" to potential buyers, who may therefore decide to stick with the imperfect strategy of hedging interest rate and stock market risk separately.

Modelling time-varying correlation and establishing corresponding traffic light option pricing formulas is thus one obvious extension of our work. Another possible extension stems from the fact that some of the traffic light options offered in practice are defined with reference interest rates other than the short rate, like, for example, longer zero coupon interest rates or swap rates. Since, in the present dynamic modelling framework these rates will be linear transformations of the short rate and thus also normally distributed random variables, extending the valuation formula accordingly should prove straightforward.

These appendices are supplied for the referee's convenience

The paper can be published with or without them

A Derivation of factor dynamics under the Q - and Q^T -measure

The basic dynamic factor system under the physical probability (\mathcal{P}) measure is assumed given as

$$dr(t) = \kappa(\theta - r(t)) dt + \sigma_r d\mathcal{W}_r(t) \quad (\text{A.1})$$

$$dS(t) = \mu S(t) dt + \sigma_S S(t) d\mathcal{W}_S(t), \quad (\text{A.2})$$

with

$$d\mathcal{W}_r(t) \cdot d\mathcal{W}_S(t) = \rho dt. \quad (\text{A.3})$$

Derivation of standard risk-neutral (Q -measure) dynamics

Define first a *money market account* by $B(t) = \exp\left\{\int_0^t r(s) ds\right\}$ such that $dB(t) = r(t)B(t) dt$, and consider next the equity portfolio value process in terms of this numeraire, i.e. $\frac{S(t)}{B(t)}$. According to the Equivalent Martingale Measure (EMM) theorem (see e.g. Hull (2006) or Duffie (2001)) we have no arbitrage iff there is a measure under which $\frac{S(t)}{B(t)}$ is a martingale. Using Ito's lemma we obtain

$$d\left(\frac{S(t)}{B(t)}\right) = \frac{1}{B(t)} dS(t) - \frac{S(t)}{B(t)^2} dB(t), \quad (\text{A.4})$$

or

$$\frac{d\left(\frac{S(t)}{B(t)}\right)}{\frac{S(t)}{B(t)}} = \frac{dS(t)}{S(t)} - \frac{dB(t)}{B(t)} = (\mu - r(t)) dt + \sigma_S d\mathcal{W}_S(t). \quad (\text{A.5})$$

Now Girsanov's theorem (see e.g. Duffie (2001)) instructs us to switch to measure Q using the substitution

$$d\mathcal{W}_S^Q(t) = d\mathcal{W}_S(t) + \lambda_S(t) dt. \quad (\text{A.6})$$

Under the Q -measure the process $\mathcal{W}_S^Q(t)$ is a standard Brownian motion and (A.5) becomes

$$\begin{aligned} \frac{d\left(\frac{S(t)}{B(t)}\right)}{\frac{S(t)}{B(t)}} &= (\mu - r(t)) dt + \sigma_S \left(d\mathcal{W}_S^Q(t) - \lambda_S(t) dt \right) \\ &= (\mu - r(t) - \sigma_S \lambda_S(t)) dt + \sigma_S d\mathcal{W}_S^Q(t). \end{aligned} \quad (\text{A.7})$$

For $\frac{S(t)}{B(t)}$ to be a Q -martingale we must require $\lambda_S(t) = \frac{\mu - r(t)}{\sigma_S}$. As a consequence, we obtain

$$\begin{aligned} dS(t) &= \mu S(t) dt + \sigma_S S(t) \left(d\mathcal{W}_S^Q(t) - \frac{\mu - r(t)}{\sigma_S} dt \right) \\ &= r(t) S(t) dt + \sigma_S S(t) d\mathcal{W}_S^Q(t). \end{aligned} \quad (\text{A.8})$$

Now consider an asset the price of which depends on the short interest rate (and time) only. Denoting this price by $P(r, t)$, Ito's lemma gives us (with subscripts on P 's denoting partial derivatives)

$$\begin{aligned} dP(r, t) &= P_t dt + P_r dr(t) + \frac{1}{2} P_{rr} (dr(t))^2 \\ &= \left(P_t + \kappa(\theta - r(t)) P_r + \frac{1}{2} \sigma_r^2 P_{rr} \right) dt + \sigma_r P_r d\mathcal{W}_r(t), \end{aligned} \quad (\text{A.9})$$

or

$$\frac{dP(r, t)}{P(r, t)} = \frac{P_t + \kappa(\theta - r(t)) P_r + \frac{1}{2} \sigma_r^2 P_{rr}}{P(r, t)} dt + \frac{\sigma_r P_r}{P(r, t)} d\mathcal{W}_r(t), \quad (\text{A.10})$$

which we will write shorter as

$$\frac{dP(r, t)}{P(r, t)} = \mu_P(r, t) dt + \sigma_P(r, t) d\mathcal{W}_r(t). \quad (\text{A.11})$$

In terms of the money market numeraire the process takes the following form

$$d\left(\frac{P(r, t)}{B(t)}\right) = \frac{1}{B(t)} dP(r, t) - \frac{P(r, t)}{B(t)^2} dB(t), \quad (\text{A.12})$$

or

$$\begin{aligned} \frac{d\left(\frac{P(r, t)}{B(t)}\right)}{\frac{P(r, t)}{B(t)}} &= \frac{dP(r, t)}{P(r, t)} - \frac{dB(t)}{B(t)} \\ &= (\mu_P(r, t) - r(t)) dt + \sigma_P(r, t) d\mathcal{W}_r(t). \end{aligned} \quad (\text{A.13})$$

As before we need to identify $\lambda_r(r, t)$ such that

$$d\mathcal{W}_r^Q(t) = d\mathcal{W}_r(t) + \lambda_r(r, t) dt, \quad (\text{A.14})$$

and such that

$$\begin{aligned} \frac{d\left(\frac{P(r,t)}{B(t)}\right)}{\frac{P(r,t)}{B(t)}} &= (\mu_P(r, t) - r(t)) dt + \sigma_P(r, t) \left(d\mathcal{W}_r^Q(t) - \lambda_r(r, t) dt \right) \\ &= (\mu_P(r, t) - r(t) - \sigma_P(r, t)\lambda_r(r, t)) dt + \sigma_P(r, t) d\mathcal{W}_r^Q(t) \end{aligned} \quad (\text{A.15})$$

is a Q -martingale. This clearly requires

$$\lambda_r(r, t) = \frac{\mu_P(r, t) - r(t)}{\sigma_P(r, t)}. \quad (\text{A.16})$$

This quantity is usually labelled the *market price of interest rate risk* and assumed constant (as for example in the term structure model of Vasicek (1977)). Here we also assume $\lambda_r(r, t)$ to be constant and henceforth write just λ_r .

As for the Q -dynamics of the interest rate process we have thus established

$$\begin{aligned} dr(t) &= \kappa(\theta - r(t)) dt + \sigma_r \left(d\mathcal{W}_r^Q(t) - \lambda_r dt \right) \\ &= \kappa\left(\theta - \frac{\sigma_r \lambda_r}{\kappa} - r(t)\right) dt + \sigma_r d\mathcal{W}_r^Q(t) \\ &= \kappa(\bar{\theta} - r(t)) dt + \sigma_r d\mathcal{W}_r^Q(t), \end{aligned} \quad (\text{A.17})$$

with $\bar{\theta} \equiv \theta - \frac{\sigma_r \lambda_r}{\kappa}$. Summing up, the Q -dynamics of our full factor system is

$$dr(t) = \kappa(\bar{\theta} - r(t)) dt + \sigma_r d\mathcal{W}_r^Q(t) \quad (\text{A.18})$$

$$dS(t) = r(t)S(t) dt + \sigma_S S(t) d\mathcal{W}_S^Q(t), \quad (\text{A.19})$$

with

$$d\mathcal{W}_r^Q(t) \cdot d\mathcal{W}_S^Q(t) = \rho dt. \quad (\text{A.20})$$

Derivation of forward-neutral (Q^T -measure) dynamics

Taking the zero-coupon bond expiring at time T as the numeraire asset instead of the money

market account will identify another martingale measure known as the forward neutral (Q^T) probability measure. We will establish the form of the factor processes under Q^T below.

From (A.10)-(A.11) and (A.15)-(A.16) the Q -dynamics of the zero-coupon bond expiring at time T is

$$\frac{dP(r, t; T)}{P(r, t; T)} = r(t) dt + \sigma_T(r, t; T) d\mathcal{W}_r^Q(t), \quad (\text{A.21})$$

where

$$\sigma_T(r, t; T) = \frac{\sigma_r P_r(r, t; T)}{P(r, t; T)}. \quad (\text{A.22})$$

Applying Ito's lemma to the P -deflated money market account we see

$$d\left(\frac{B(t)}{P(r, t; T)}\right) = \frac{1}{P(r, t; T)} dB(t) - \frac{B(t)}{P^2(r, t; T)} dP(r, t; T) + \frac{B(t)}{P^3(r, t; T)} (dP(r, t; T))^2, \quad (\text{A.23})$$

or

$$\begin{aligned} \frac{d\left(\frac{B(t)}{P(r, t; T)}\right)}{\frac{B(t)}{P(r, t; T)}} &= \frac{dB(t)}{B(t)} - \frac{dP(r, t; T)}{P(r, t; T)} + \left(\frac{dP(r, t; T)}{P(r, t; T)}\right)^2 \\ &= \sigma_T^2(r, t; T) dt - \sigma_T(r, t; T) d\mathcal{W}_r^Q(t). \end{aligned} \quad (\text{A.24})$$

Using the Girsanov and the EMM theorems as before we set

$$d\mathcal{W}_r^{Q^T}(t) = d\mathcal{W}_r^Q(t) + \lambda_r^T(t) dt, \quad (\text{A.25})$$

and

$$\begin{aligned} \frac{d\left(\frac{B(t)}{P(r, t; T)}\right)}{\frac{B(t)}{P(r, t; T)}} &= \sigma_T^2(r, t; T) dt - \sigma_T(r, t; T) \left(d\mathcal{W}_r^{Q^T}(t) - \lambda_r^T(t) dt\right) \\ &= \sigma_T(r, t; T) (\sigma_T(r, t; T) + \lambda_r^T(t)) dt - \sigma_T(r, t; T) d\mathcal{W}_r^{Q^T}(t), \end{aligned} \quad (\text{A.26})$$

which is a Q^T -martingale iff

$$\lambda_r^T(t) = -\sigma_T(r, t; T). \quad (\text{A.27})$$

Now consider the equity portfolio value deflated by $P(r, t; T)$:

$$\begin{aligned} d\left(\frac{S(t)}{P(r, t; T)}\right) &= \frac{1}{P(r, t; T)} dS(t) - \frac{S(t)}{P^2(r, t; T)} dP(r, t; T) + \frac{S(t)}{P^3(r, t; T)} (dP(r, t; T))^2 \\ &\quad - \frac{1}{P^2(r, t; T)} dS(t) dP(r, t; T), \end{aligned} \quad (\text{A.28})$$

or

$$\begin{aligned}
\frac{d\left(\frac{S(t)}{P(r,t;T)}\right)}{\frac{S(t)}{P(r,t;T)}} &= \frac{dS(t)}{S(t)} - \frac{dP(r,t;T)}{P(r,t;T)} + \left(\frac{dP(r,t;T)}{P(r,t;T)}\right)^2 - \frac{dS(t)}{S(t)} \frac{dP(r,t;T)}{P(r,t;T)} \\
&= (\sigma_T^2(r,t;T) - \rho\sigma_S\sigma_T(r,t;T)) dt + \sigma_S d\mathcal{W}_S^Q(t) - \sigma_T(r,t;T) d\mathcal{W}_r^Q(t).
\end{aligned} \tag{A.29}$$

Using (A.25), (A.27), and letting

$$d\mathcal{W}_S^{Q^T}(t) = d\mathcal{W}_S^Q(t) + \lambda_S^T(t) dt, \tag{A.30}$$

we get

$$\begin{aligned}
\frac{d\left(\frac{S(t)}{P(r,t;T)}\right)}{\frac{S(t)}{P(r,t;T)}} &= (\sigma_T^2(r,t;T) - \rho\sigma_S\sigma_T(r,t;T)) dt + \sigma_S \left(d\mathcal{W}_S^{Q^T}(t) - \lambda_S^T(t) dt\right) \\
&\quad - \sigma_T(r,t;T) \left(d\mathcal{W}_r^{Q^T}(t) + \sigma_T(r,t;T) dt\right) \\
&= (-\rho\sigma_S\sigma_T(r,t;T) - \sigma_S\lambda_S^T(t)) dt + \sigma_S d\mathcal{W}_S^{Q^T}(t) - \sigma_T(r,t;T) d\mathcal{W}_r^{Q^T}(t).
\end{aligned} \tag{A.31}$$

To obtain a Q^T -martingale here we clearly require

$$\lambda_S^T(t) = -\rho\sigma_T(r,t;T). \tag{A.32}$$

We can now substitute in (A.18) and (A.19) to obtain the forward neutral factor dynamics:

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= r(t) dt + \sigma_S \left(d\mathcal{W}_S^{Q^T}(t) - \lambda_S^T(t) dt\right) \\
&= (r(t) + \rho\sigma_S\sigma_T(r,t;T)) dt + \sigma_S d\mathcal{W}_S^{Q^T}(t),
\end{aligned} \tag{A.33}$$

and

$$\begin{aligned}
dr(t) &= \kappa(\bar{\theta} - r(t)) dt + \sigma_r \left(d\mathcal{W}_r^{Q^T}(t) - \lambda_r^T(t) dt\right) \\
&= \left(\kappa(\bar{\theta} - r(t)) + \sigma_r\sigma_T(r,t;T)\right) dt + \sigma_r d\mathcal{W}_r^{Q^T}(t).
\end{aligned} \tag{A.34}$$

Lastly we are able to identify the exact form of $\sigma_T(r,t;T)$ since from (A.10) we know that

$$\sigma_T(r,t;T) = \frac{\sigma_r P_r(r,t;T)}{P(r,t;T)}, \tag{A.35}$$

and since an explicit solution for zero-coupon bond prices is well-known in the present setup. From this expression (see (11) in the paper's main text) we see that

$$P_r(r, t; T) = -\Psi(T - t)P(r, t; T), \quad (\text{A.36})$$

and therefore have

$$\sigma_T(r, t; T) = -\sigma_r \Psi(T - t). \quad (\text{A.37})$$

The Q^T -factor dynamics thus take the following form:

$$dr(t) = \left(\kappa(\bar{\theta} - r(t)) - \sigma_r^2 \Psi(T - t) \right) dt + \sigma_r d\mathcal{W}_r^{Q^T}(t) \quad (\text{A.38})$$

$$\frac{dS(t)}{S(t)} = \left(r(t) - \rho \sigma_S \sigma_r \Psi(T - t) \right) dt + \sigma_S d\mathcal{W}_S^{Q^T}(t), \quad (\text{A.39})$$

with

$$d\mathcal{W}_r^{Q^T}(t) \cdot d\mathcal{W}_S^{Q^T}(t) = \rho dt, \quad (\text{A.40})$$

and

$$\Psi(x) = \frac{1 - e^{-\kappa x}}{\kappa}. \quad (\text{A.41})$$

B Derivation of the joint Q^T -distribution of $r(T)$ and $S(T)$.

The explicit interest rate process under Q^T

The stochastic differential equation for the interest rate derived in (A.38) implies the following explicit solution (see Arnold (1992)) for $t < s \leq T$:

$$\begin{aligned} r(s) &= r(t)e^{-\kappa(s-t)} + \left(\bar{\theta} - \frac{\sigma_r^2}{\kappa^2} \right) \left(1 - e^{-\kappa(s-t)} \right) \\ &\quad + \frac{\sigma_r^2}{2\kappa^2} e^{-\kappa(T-s)} \left(1 - e^{-2\kappa(s-t)} \right) + \sigma_r \int_t^s e^{-\kappa(s-u)} d\mathcal{W}_r^{Q^T}(u). \end{aligned} \quad (\text{B.1})$$

We will also be needing the interest rate integral $\int_t^T r(s) ds$. Straightforward but tedious calculations show that this can be represented as

$$\begin{aligned} \int_t^T r(s) ds &= \left(\bar{\theta} - \frac{\sigma_r^2}{\kappa^2} \right) (T-t) + \left(r(t) - \bar{\theta} + \frac{\sigma_r^2}{\kappa^2} \right) \Psi(T-t) \\ &\quad + \frac{\sigma_r^2}{2\kappa} \Psi^2(T-t) + \sigma_r \int_t^T \Psi(T-u) d\mathcal{W}_r^{Q^T}(u), \end{aligned} \quad (\text{B.2})$$

where again $\Psi(x) = \frac{1-e^{-\kappa x}}{\kappa}$.

The explicit $\ln S(T)$ -process under Q^T

From (A.39) and Ito's lemma we get

$$\begin{aligned} d(\ln S(t)) &= \frac{dS(t)}{S(t)} - \frac{1}{2} \left(\frac{dS(t)}{S(t)} \right)^2 \\ &= \left(r(t) - \rho\sigma_S\sigma_r\Psi(T-t) - \frac{1}{2}\sigma_S^2 \right) dt + \sigma_S d\mathcal{W}_S^{Q^T}(t), \end{aligned} \quad (\text{B.3})$$

and hence

$$\begin{aligned} \ln S(T) &= \ln S(t) + \int_t^T \left(r(s) - \rho\sigma_S\sigma_r\Psi(T-s) - \frac{1}{2}\sigma_S^2 \right) ds + \sigma_S \int_t^T d\mathcal{W}_S^{Q^T}(s) \\ &= \ln S(t) + \int_t^T r(s) ds - \rho\sigma_S\sigma_r \int_t^T \Psi(T-s) ds - \frac{1}{2}\sigma_S^2(T-t) + \sigma_S \int_t^T d\mathcal{W}_S^{Q^T}(s). \end{aligned} \quad (\text{B.4})$$

Substituting (B.2) for the interest rate integral in (B.4) and integrating the Ψ -function yields

$$\begin{aligned}
\ln S(T) &= \ln S(t) \\
&+ \left(\bar{\theta} - \frac{\sigma_r^2}{\kappa^2} \right) (T - t) \\
&+ \left(r(t) - \bar{\theta} + \frac{\sigma_r^2}{\kappa^2} \right) \Psi(T - t) \\
&+ \frac{\sigma_r^2}{2\kappa} \Psi^2(T - t) \\
&- \frac{\rho\sigma_S\sigma_r}{\kappa} \left((T - t) - \Psi(T - t) \right) \\
&- \frac{1}{2}\sigma_S^2(T - t) \\
&+ \frac{\sigma_r}{\kappa} \int_t^T \left(1 - e^{-\kappa(T-u)} \right) d\mathcal{W}_r^{Q^T}(u) \\
&+ \sigma_S \int_t^T d\mathcal{W}_S^{Q^T}(s) \\
&= \ln S(t) \\
&+ \left(\bar{\theta} - \frac{\sigma_r^2}{\kappa^2} - \frac{\rho\sigma_S\sigma_r}{\kappa} - \frac{1}{2}\sigma_S^2 \right) (T - t) \\
&- \left(\bar{\theta} - \frac{\sigma_r^2}{\kappa^2} - \frac{\rho\sigma_S\sigma_r}{\kappa} - r(t) \right) \Psi(T - t) \\
&+ \frac{\sigma_r^2}{2\kappa} \Psi^2(T - t) \\
&+ \sigma_r \int_t^T \Psi(T - u) d\mathcal{W}_r^{Q^T}(u) \\
&+ \sigma_S \int_t^T d\mathcal{W}_S^{Q^T}(s).
\end{aligned} \tag{B.5}$$

It follows from (B.1) and (B.4) that the joint conditional Q^T -distribution of $r(T)$ and $\ln S(T)$ is a bivariate normal distribution. Below we provide the conditional means, variances, and the covariance expression relating to this distribution.

From (B.1) and (B.4) we establish directly the following (conditional) means:

$$E_t^{Q^T} \{r(T)\} = r(t) + \kappa(\bar{\theta} - r(t))\Psi(T-t) - \frac{\sigma_r^2}{2}\Psi^2(T-t) \quad (\text{B.6})$$

$$\begin{aligned} E_t^{Q^T} \{\ln S(T)\} &= \ln S(t) + \left(\bar{\theta} - \frac{\sigma_r^2}{\kappa^2} - \frac{\rho\sigma_S\sigma_r}{\kappa} - \frac{1}{2}\sigma_S^2 \right) (T-t) \\ &\quad - \left(\bar{\theta} - \frac{\sigma_r^2}{\kappa^2} - \frac{\rho\sigma_S\sigma_r}{\kappa} - r(t) \right) \Psi(T-t) + \frac{\sigma_r^2}{2\kappa}\Psi^2(T-t). \end{aligned} \quad (\text{B.7})$$

The conditional Q^T -variance of the interest rate is easily established as

$$\text{Var}_t^{Q^T} \{r(T)\} = \frac{\sigma_r^2}{2\kappa} \left(1 - e^{-2\kappa(T-t)} \right). \quad (\text{B.8})$$

As for the conditional Q^T -variance of $\ln S(T)$ we have

$$\begin{aligned} \text{Var}_t^{Q^T} \{\ln S(T)\} &= \text{Var}_t^{Q^T} \left\{ \sigma_r \int_t^T \Psi(T-u) d\mathcal{W}_r^{Q^T}(u) + \sigma_S \int_t^T d\mathcal{W}_S^{Q^T}(s) \right\} \\ &= \sigma_r^2 \int_t^T \Psi^2(T-u) du + \sigma_S^2 \int_t^T ds \\ &\quad + 2\sigma_r\sigma_S \text{Cov}_t^{Q^T} \left\{ \int_t^T \Psi(T-u) d\mathcal{W}_r^{Q^T}(u), \int_t^T d\mathcal{W}_S^{Q^T}(s) \right\} \\ &= \vdots \\ &= (T-t) \left(\frac{\sigma_r^2}{\kappa^2} + \sigma_S^2 + \frac{2\rho\sigma_r\sigma_S}{\kappa} \right) \\ &\quad - \Psi(T-t) \left(\frac{\sigma_r^2}{\kappa^2} + \frac{2\rho\sigma_r\sigma_S}{\kappa} \right) \\ &\quad - \Psi^2(T-t) \frac{\sigma_r^2}{2\kappa}. \end{aligned} \quad (\text{B.9})$$

Finally we establish the conditional covariance:

$$\begin{aligned} \text{Cov}_t^{Q^T} \{r(T), \ln S(T)\} &= \text{Cov}_t^{Q^T} \left\{ \sigma_r \int_t^T \Psi(T-u) d\mathcal{W}_r^{Q^T}(u) + \sigma_S \int_t^T d\mathcal{W}_S^{Q^T}(s), \right. \\ &\quad \left. \sigma_r \int_t^T e^{-\kappa(T-u)} d\mathcal{W}_r^{Q^T}(u) \right\} \\ &= \sigma_S\sigma_r \int_t^T e^{-\kappa(T-u)} \rho du + \sigma_r^2 \int_t^T \Psi(T-u) e^{-\kappa(T-u)} du \\ &= \vdots \\ &= \rho\sigma_S\sigma_r\Psi(T-t) + \frac{\sigma_r^2}{2}\Psi^2(T-t). \end{aligned} \quad (\text{B.10})$$

C Some Truncated Means in the Bivariate Normal Distribution

Basics of the bivariate normal distribution

Suppose $\begin{pmatrix} x \\ y \end{pmatrix}$ is bivariate normal (see e.g. Kotz, Balakrishnan, and Johnson (2000)), i.e.

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim N \left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} \right), \quad (\text{C.1})$$

where $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2$ and σ_{xy} are the constant mean, variance, and covariance coefficients, respectively. The coefficient of correlation is given as

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}.$$

The corresponding density function, $f(x, y)$, is given as

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \exp \left\{ \frac{-1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_x}{\sigma_x} \right)^2 - 2\rho \left(\frac{x-\mu_x}{\sigma_x} \right) \left(\frac{y-\mu_y}{\sigma_y} \right) + \left(\frac{y-\mu_y}{\sigma_y} \right)^2 \right] \right\}. \quad (\text{C.2})$$

The bivariate density can be factorized as

$$f(x, y) = f(x) \cdot f(y|x), \quad (\text{C.3})$$

where $f(x)$ is the marginal (normal) density of x ,

$$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left(\frac{x-\mu_x}{\sigma_x} \right)^2 \right\}, \quad (\text{C.4})$$

and where $f(y|x)$ is the density of y conditional on x :

$$f(y|x) = \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \exp \left\{ \frac{-1}{2\sigma_y^2(1-\rho^2)} \left(y - \mu_y - \frac{\rho\sigma_y}{\sigma_x}(x - \mu_x) \right)^2 \right\}. \quad (\text{C.5})$$

The conditional density of y given x is thus 1-dimensional normal:

$$y|x \sim N \left(\mu_y + \frac{\rho\sigma_y}{\sigma_x}(x - \mu_x), \sigma_y^2(1 - \rho^2) \right). \quad (\text{C.6})$$

The fundamental problem

Valuation of the traffic light option boils down to calculating the following expectation:

$$\begin{aligned} E \{ [\bar{x} - x]^+ \cdot [\bar{Y} - Y]^+ \} &\equiv E \{ [\bar{x} - x]^+ \cdot [\bar{Y} - e^y]^+ \} \\ &= E \{ [\bar{x} - x] \cdot [\bar{Y} - e^y] \cdot 1_{x < \bar{x}} \cdot 1_{y < \bar{y}} \}, \end{aligned} \quad (\text{C.7})$$

where $Y \equiv e^y$, $\bar{y} \equiv \ln \bar{Y}$, and where $\begin{pmatrix} x \\ y \end{pmatrix}$ is bivariate normal. Using linearity of the mean, (C.7) can be decomposed as follows:

$$\begin{aligned} E \{ [\bar{x} - x] \cdot [\bar{Y} - e^y] \cdot 1_{x < \bar{x}} \cdot 1_{y < \bar{y}} \} &= \bar{x} \cdot \bar{Y} \cdot E \{ 1_{x < \bar{x}} \cdot 1_{y < \bar{y}} \} - \bar{x} \cdot E \{ e^y \cdot 1_{x < \bar{x}} \cdot 1_{y < \bar{y}} \} \\ &\quad - \bar{Y} \cdot E \{ x \cdot 1_{x < \bar{x}} \cdot 1_{y < \bar{y}} \} + E \{ x \cdot e^y \cdot 1_{x < \bar{x}} \cdot 1_{y < \bar{y}} \}. \end{aligned}$$

Below we consider each of these four right-hand side expectation terms in turn.

The first expectation term

We must first calculate

$$\begin{aligned} E \{ 1_{x < \bar{x}} \cdot 1_{y < \bar{y}} \} &= \int_{-\infty}^{\bar{x}} \int_{-\infty}^{\bar{y}} f(x, y) dy dx \\ &= \int_{-\infty}^{\bar{x}} \int_{-\infty}^{\bar{y}} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \\ &\quad \exp \left\{ \frac{-1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_x}{\sigma_x} \right)^2 - 2\rho \left(\frac{x-\mu_x}{\sigma_x} \right) \left(\frac{y-\mu_y}{\sigma_y} \right) + \left(\frac{y-\mu_y}{\sigma_y} \right)^2 \right] \right\} dy dx. \end{aligned}$$

Changing variables

$$u = \frac{x - \mu_x}{\sigma_x}, \quad v = \frac{y - \mu_y}{\sigma_y},$$

we get

$$\begin{aligned} E \{ 1_{x < \bar{x}} \cdot 1_{y < \bar{y}} \} &= \int_{-\infty}^{\bar{u}} \int_{-\infty}^{\bar{v}} \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \exp \left\{ \frac{-1}{2(1-\rho^2)} [u^2 - 2\rho uv + v^2] \right\} dv du \\ &= M(\bar{u}, \bar{v}; \rho) \\ &= M \left(\frac{\bar{x} - \mu_x}{\sigma_x}, \frac{\bar{y} - \mu_y}{\sigma_y}; \rho \right), \end{aligned} \quad (\text{C.8})$$

where $M(\cdot, \cdot, \rho)$ denotes the cumulative probability in the standardized bivariate normal distribution with correlation coefficient ρ .

The second expectation term

Here we need to calculate

$$E \{ e^y \cdot 1_{x < \bar{x}} \cdot 1_{y < \bar{y}} \} = \int_{-\infty}^{\bar{x}} \int_{-\infty}^{\bar{y}} e^y \cdot f(x, y) dy dx.$$

We first make use of the factorization (C.3)-(C.5) and consider the exponent

$$y - \frac{1}{2} \left(\frac{x - \mu_x}{\sigma_x} \right)^2 - \frac{1}{2\sigma_y^2(1 - \rho^2)} \left(y - \mu_y - \frac{\rho\sigma_y}{\sigma_x} (x - \mu_x) \right)^2. \quad (\text{C.9})$$

Changing variables as before

$$u = \frac{x - \mu_x}{\sigma_x}, \quad v = \frac{y - \mu_y}{\sigma_y},$$

we can rewrite (C.9) as follows;

$$\begin{aligned} & \mu_y + v\sigma_y - \frac{1}{2}u^2 - \frac{1}{2(1 - \rho^2)}(v - \rho u)^2 \\ = & \mu_y - \frac{1}{2(1 - \rho^2)}(v^2 + \rho^2 u^2 - 2\rho uv - 2\sigma_y v(1 - \rho^2) + u^2(1 - \rho^2)) \\ = & \mu_y - \frac{1}{2(1 - \rho^2)}(v^2 + u^2 - 2\rho uv - 2\sigma_y v(1 - \rho^2)). \end{aligned} \quad (\text{C.10})$$

Making another variable change;

$$w = u - \rho\sigma_y, \quad z = v - \sigma_y,$$

relation (C.10) becomes

$$\begin{aligned} & \mu_y - \frac{1}{2(1 - \rho^2)}(w^2 + z^2 - 2\rho w z - \sigma_y^2(1 - \rho^2)) \\ = & \mu_y + \frac{1}{2}\sigma_y^2 - \frac{1}{2(1 - \rho^2)}(w^2 + z^2 - 2\rho w z). \end{aligned} \quad (\text{C.11})$$

We therefore have

$$\begin{aligned} E \{ e^y \cdot 1_{x < \bar{x}} \cdot 1_{y < \bar{y}} \} &= \int_{-\infty}^{\bar{x}} \int_{-\infty}^{\bar{y}} e^y \cdot f(x, y) dy dx \\ &= e^{\mu_y + \frac{1}{2}\sigma_y^2} \int_{-\infty}^{\bar{w}} \int_{-\infty}^{\bar{z}} \frac{1}{2\pi\sqrt{1 - \rho^2}} \times \\ & \quad \exp \left\{ \frac{-1}{2(1 - \rho^2)}(w^2 - 2\rho w z + z^2) \right\} dz dw \\ &= e^{\mu_y + \frac{1}{2}\sigma_y^2} \cdot M \left(\frac{\bar{x} - \mu_x}{\sigma_x} - \rho\sigma_y, \frac{\bar{y} - \mu_y}{\sigma_y} - \sigma_y; \rho \right). \end{aligned} \quad (\text{C.12})$$

The third expectation term

In this case we need to calculate

$$E \{x \cdot 1_{x < \bar{x}} \cdot 1_{y < \bar{y}}\} = \int_{-\infty}^{\bar{x}} \int_{-\infty}^{\bar{y}} x \cdot f(x, y) dy dx. \quad (\text{C.13})$$

Our usual change of variables $u = \frac{x - \mu_x}{\sigma_x}$, and $v = \frac{y - \mu_y}{\sigma_y}$ yields

$$\begin{aligned} & \int_{-\infty}^{\bar{u}} \int_{-\infty}^{\bar{v}} (\mu_x + \sigma_x u) \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left\{ \frac{-1}{2(1-\rho^2)} (u^2 - 2\rho uv + v^2) \right\} dv du \\ = & \mu_x \cdot M \left(\frac{\bar{x} - \mu_x}{\sigma_x}, \frac{\bar{y} - \mu_y}{\sigma_y}; \rho \right) \\ & + \sigma_x \int_{-\infty}^{\bar{u}} \int_{-\infty}^{\bar{v}} u \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left\{ \frac{-1}{2(1-\rho^2)} (u^2 - 2\rho uv + v^2) \right\} dv du. \quad (\text{C.14}) \end{aligned}$$

Using the factorization (C.3)-(C.5) and letting $n(\cdot)$ denote the density of the standard normal distribution, the double integral in (C.14) can be manipulated as follows:

$$\sigma_x \int_{-\infty}^{\bar{u}} u \cdot n(u) \int_{-\infty}^{\bar{v}} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} (v - \rho u)^2 \right\} dv du. \quad (\text{C.15})$$

Defining now $w = \frac{v - \rho u}{\sqrt{1-\rho^2}}$ (C.15) becomes

$$\begin{aligned} & \sigma_x \int_{-\infty}^{\bar{u}} u \cdot n(u) \int_{-\infty}^{\bar{w}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} dw du \\ = & \sigma_x \int_{-\infty}^{\bar{u}} u \cdot n(u) \cdot N(\bar{w}) du \\ = & \sigma_x E \left\{ \tilde{\varepsilon} \cdot N \left(\frac{\bar{v} - \rho \tilde{\varepsilon}}{\sqrt{1-\rho^2}} \right) \cdot 1_{\tilde{\varepsilon} < \bar{u}} \right\}, \quad (\text{C.16}) \end{aligned}$$

where $\tilde{\varepsilon} \sim N(0, 1)$, and where $N(\cdot)$ denotes the cumulative probability of the standard normal distribution.

The full expression is thus

$$\begin{aligned} E \{x \cdot 1_{x < \bar{x}} \cdot 1_{y < \bar{y}}\} = & \mu_x M \left(\frac{\bar{x} - \mu_x}{\sigma_x}, \frac{\bar{y} - \mu_y}{\sigma_y}; \rho \right) \\ & + \sigma_x E \left\{ \tilde{\varepsilon} \cdot N \left(\frac{\frac{\bar{y} - \mu_y}{\sigma_y} - \rho \tilde{\varepsilon}}{\sqrt{1-\rho^2}} \right) \cdot 1_{\tilde{\varepsilon} < \frac{\bar{x} - \mu_x}{\sigma_x}} \right\}. \quad (\text{C.17}) \end{aligned}$$

The fourth expectation term

Finally we must calculate

$$E \{x \cdot e^y \cdot 1_{x < \bar{x}} \cdot 1_{y < \bar{y}}\} = \int_{-\infty}^{\bar{x}} \int_{-\infty}^{\bar{y}} x \cdot e^y \cdot f(x, y) dy dx. \quad (\text{C.18})$$

Drawing on our calculations in (C.9)-(C.11) we first rewrite the double integral;

$$\begin{aligned} & \int_{-\infty}^{\bar{x}} \int_{-\infty}^{\bar{y}} x \cdot e^y \cdot f(x, y) dy dx \\ = & e^{\mu_y + \frac{1}{2}\sigma_y^2} \int_{-\infty}^{\bar{w}} \int_{-\infty}^{\bar{z}} (\mu_x + \rho\sigma_x\sigma_y + \sigma_x w) \frac{1}{2\pi\sqrt{1-\rho^2}} \times \\ & \exp\left\{-\frac{1}{2(1-\rho^2)}(w^2 + z^2 - 2\rho wz)\right\} dz dw \\ = & (\mu_x + \rho\sigma_x\sigma_y) e^{\mu_y + \frac{1}{2}\sigma_y^2} \cdot M(\bar{w}, \bar{z}; \rho) \\ & + \sigma_x e^{\mu_y + \frac{1}{2}\sigma_y^2} \int_{-\infty}^{\bar{w}} \int_{-\infty}^{\bar{z}} w \cdot \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(w^2 + z^2 - 2\rho wz)\right\} dz dw, \end{aligned}$$

where we have made the substitutions

$$w = \frac{x - \mu_x}{\sigma_x} - \rho\sigma_y, \quad z = \frac{y - \mu_y}{\sigma_y} - \sigma_y.$$

The double integral in the second term above is now identical in form to the term considered previously in (C.14)-(C.16) so we conclude

$$\begin{aligned} E \{x \cdot e^y \cdot 1_{x < \bar{x}} \cdot 1_{y < \bar{y}}\} = & (\mu_x + \rho\sigma_x\sigma_y) e^{\mu_y + \frac{1}{2}\sigma_y^2} \cdot M\left(\frac{\bar{x} - \mu_x}{\sigma_x} - \rho\sigma_y, \frac{\bar{y} - \mu_y}{\sigma_y} - \sigma_y; \rho\right) \\ & + \sigma_x e^{\mu_y + \frac{1}{2}\sigma_y^2} \cdot E \left\{ \tilde{\varepsilon} \cdot N\left(\frac{\frac{\bar{y} - \mu_y}{\sigma_y} - \sigma_y - \rho\tilde{\varepsilon}}{\sqrt{1-\rho^2}}\right) \cdot 1_{\tilde{\varepsilon} < \frac{\bar{x} - \mu_x}{\sigma_x} - \rho\sigma_y} \right\}, \end{aligned} \quad (\text{C.19})$$

where $\tilde{\varepsilon} \sim N(0, 1)$.

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