Calling Convertible Bonds too Late Can be Rational

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Abstract

In this paper, we offer a theoretical explanation for the empirical puzzle that convertible bonds are called too late or too early. Contrary to the literature, we do not assume frictions to explain the observed call behavior. Instead, we assume that the firm has issued in addition to stock and convertible bonds, also straight bonds. This seemingly innocuous extension has the important consequence that a conversion can cause a wealth transfer from the stockholders to the bondholders. This wealth transfer changes the analysis and results of the seminal papers by Ingersoll (1977a) and Brennan/Schwartz (1977) substantially. First, the value of a convertible bond can no longer be determined by option-pricing methods. The strategic behavior of the convertible bond holders requires a game-theoretical analysis. Second, the negative effect of an enforced conversion on the stockholders can be avoided by calling the bonds too late. Third, a positive wealth transfer can also be produced by calling the bond too early.

JEL-Classification: G12, G13, G32

1 Introduction

Ingersoll (1977a) and Brennan/Schwartz (1977) showed in their seminal papers that it is optimal to call a convertible bond whenever the value of the convertible bond

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exceeds the call price and not to call otherwise. The majority of valuation models for callable convertible bonds makes use of this result. This call policy is easy to implement and also has the advantage that its test only rests on observable market prices. In the following we denote this policy as *IBS strategy*.

Already Ingersoll (1977b) pointed out that firms do not follow the IBS strategy. They call their convertible bond issues compared to the prediction of IBS too late. These findings are confirmed by many other studies for callable convertible and non-convertible bonds (see e.g. Mikkelson (1981), Vu (1986), Constantinides/Grundy (1987), Asquith/Mullins (1991), and King/Mauer (2000)). To reconcile the theoretical predictions with the observed call behavior, frictions were introduced to the models. Typical frictions to explain late calls are: a notice period of positive length between the call announcement and the call execution (see e.g. Butler (2002) and Kwok/Wu (2000)), transaction costs for refunding a called convertible bond issue (see Mauer (1993)), the tax deductability arising from coupon payments of the convertible bond (see e.g. Asquith/Mullins (1991) and Jalan/Barone-Adesi (1995)), and "sleeping investors" who delay an optimal voluntary conversion (see e.g. Dunn/Eades (1989)). Constantinides/Grundy (1987) present a general overview of possible frictions.

The goal of this paper is to provide a theoretical explanation for the late call puzzle. It also offers an explanation for calls that are too early relative to the IBS strategy as reported e.g. by Asquith (1995). Contrary to the existing literature, we will argue within an ideal Modigiliani/Miller world without any frictions. The only extension relative to the approaches by Brennan/Schwartz (1977) and Ingersoll (1977a) is that we allow for a more *general capital structure*. We consider firms that have issued in addition to stocks and convertible bonds, also straight bonds. This seemingly innocuous generalization has an important consequence. A conversion increases the equity and reduces the credit risk for the additional bondholders at the expense of the stockholders. This *wealth transfer* from stockholders to the bondholders changes the analysis and the results of Ingersoll (1977a) and Brennan/Schwartz (1977) substantially.

First, optimal conversion strategies and the value of a convertible bond can no longer be characterized by option-pricing methods, because the strategic behavior of the convertible bond holders matters. Contrary to the results by Constantinides (1984) and Emanuel (1983), Bühler/Koziol (2002) show that in a competitive market setting, the value of a convertible bond in the unique Nash equilibrium generally differs from the value obtained by option-pricing approaches.

Second, it can be optimal to call a convertible bond *too late* relative to the IBS strategy. The reason is that a call possibly enforces a conversion which hurts the stockholders because of the wealth transfer to the additional bondholders.

Third and surprisingly, it can be optimal to call a convertible bond too early relative to the IBS strategy, i.e. if the conversion value and the value of the convertible bond are below the call price. This is the case if the call decision hurts the additional bondholders in favor of the stockholders. As a consequence of this case, the value of a callable convertible bond can be above the value of an identical but not callable convertible bond.

Summing up, we argue that there is no obvious empirical puzzle in the observed call behavior of convertible bonds. As most firms have additional debt outstanding, their call policy can be in line with a model that considers the strategic behavior of convertible bond holders and the induced wealth transfer between stockholders and debt holders.

The paper is organized as follows. Section 2 describes the model framework and shows how to price callable convertible bonds, stocks, and additional debt for a given conversion and call strategy. In Section 3, we present the optimal conversion decision at maturity and the optimal conversion volume enforced by a call before maturity. The optimal call policy of the firm and the values of both a stock and a callable convertible bond in equilibrium are derived in Section 4. Section 5 concludes.

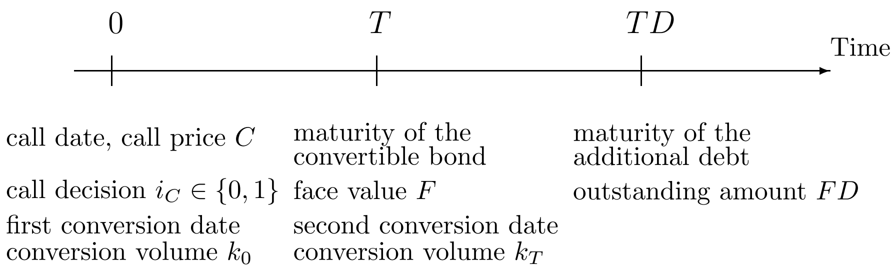
2 Model Framework and Asset Values

2.1 Framework

To study the effects of additional debt on the optimal call policy for convertible bonds, we consider the most simple setup with regard to the capital structure of the firm, the terms of the bonds, and the possibilities to call and to convert the convertible bonds. Extensions which are necessary to price quoted convertible bonds are straightforward and will be discussed in Section 5.

The following analysis refers to a firm which has *outstanding* n stocks, m callable convertible bonds, and additional debt in form of a straight bond. Both bonds, the convertible and the straight bond, are assumed to be *zero-coupon* bonds. The face value of an individual convertible bond is denoted by F and the total nominal value of additional debt by FD. The convertible bond matures at time T which is

Figure 1: Time Structure of the Model



before the maturity TD of the additional debt. Therefore, the firm is levered over the whole lifetime of the convertible bond. Each convertible bond can be converted into *one* newly-issued stock upon the notice of its holder. The additional debt is subordinated to the convertible bonds. The assumptions regarding the priority rule in the case of bankruptcy can be easily modified without changing the basic results. E.g., both bond issues could be subject to reverse or equal priority rules. The maturity assumption $T < TD$ is essential for the following results.

All stocks, the n initially outstanding and those generated by a conversion, have identical rights and are non-dividend paying. This assumption implies that a conversion is only optimal at maturity or when it is *enforced* by a call.

Coupon paying bonds issued on capital markets in Europe can be typically called only at coupon dates. Therefore, it is sufficient to consider a finite number of call dates and enforced conversion possibilities. In the U.S.-American bond markets a call can usually be announced at any time after the call protection period. For simplicity, we assume the typical European call structure with one call possibility and a notice period of zero length. The recursive numerical extensions to cover several call dates and a non-zero notice period are straightforward.

The time structure of our model is presented in Figure 1. At date $t = 0$, the firm, that is assumed to act in the interest of the old stockholders, decides whether the convertible bonds will be called ($i_C = 1$). Afterwards the convertible bond holders decide about the number k_0 ($0 \leq k_0 \leq m$) of bonds they convert. If the call option is exercised, the remaining $m - k_0$ bonds will be redeemed at the call price C.[1] A second conversion possibility exists at the maturity date T of the convertible bond. Here the conversion volume k_T is restricted to $0 \leq k_T \leq m - k_0$.

We assume that a default of the firm can only occur if an obligatory expense is not covered by the firm value V_t at the same time. This definition implies that a default is possible only at the dates $t = 0$, $t = T$ or $t = TD$. At the first date the firm defaults if it calls the convertible bonds and the redemption volume $(m - k_0) \cdot C$ is larger than V_0 before redemption. At the maturity date T of the convertible bonds, the firm defaults if the redemption value $(m - k_0 - k_T) \cdot F$ is larger than the firm value V_T before redemption. Finally, the firm defaults at $t = TD$ if the firm value is below the face value FD of the additional debt. Obviously, the default probability and the loss given default depend on the call and conversion strategies.

If the firm defaults, it will be liquidated and the proceeds, which are assumed to equal the firm value before default, are distributed according to the priority rule. This rule implies that at points of time 0 and T only the convertible bond holders participate in the firm value.

The state variable in our model is the value V_t of an identical, unlevered firm.[2] V_t is assumed to follow a geometric Brownian motion

$$dV_t = \mu \cdot V_t \cdot dt + \sigma \cdot V_t \cdot dz_t \qquad (t < TD; t \neq 0, T)\,. \tag{1}$$

z_t denotes the value of the standard Wiener process, $\mu > 0$ and $\sigma > 0$ are the mean and volatility of the firm's instantaneous return. Depending on the call and conversion strategy, the firm has to cover expenses at $t = 0, T$. We impose two standard assumptions. First, we assume that the payments $(m - k_0) \cdot C$ and $(m - k_0 - k_T) \cdot F$ are covered by selling assets of the same value. Second, we assume that the reduction in the firm value affects the volume but not the return structure of the firm's investment program. This scale-reduction approach justifies that the drift and diffusion coefficient in (1) is independent of the call strategy and the conversion volume.

In the following, we denote by V_0^- and V_T^- the (existing) left limits of V_t at $t = 0$ and $t = T$. Then, the relation between the firm value V_0^- before and V_0 after the call decision i_C and the conversion decision k_0 is given by

$$V_0 = \max\left\{V_0^- - i_C \cdot (m - k_0) \cdot C, 0\right\}. \tag{2}$$

[1] In this expository section, we allow for non-optimal call and conversion decisions.

[2] As we argue in a Modigiliani/Miller framework it is not contradictory to assume that an identical, unleverd firm exists.

The decision variable i_C is equal to one in the case of a call and zero otherwise.

Analogously, at maturity the relation between the firm values V_T^-, V_T before and after redemption is given by

$$V_T = \max\left\{V_T^- - (m - k_0 - k_T) \cdot F, 0\right\}. \tag{3}$$

All outstanding securities of the firm, stocks, convertible bonds, and additional bonds (debt), trade continuously on competitive, complete, and frictionless markets without information asymmetries. In addition, *default-free* zero bonds with arbitrary maturities and a constant yield to maturity r are also traded. Finally, we assume that there is a continuum of convertible bond holders. They act competitively when making their conversion decision. This conversion variant is called *unrestricted* or *competitive conversion*. Constantinides (1984) showed that in this model setting the value of a convertible bond in the "best" Nash equilibrium equals the value if a *block conversion* is assumed. This type of strategy is implicitly used for the option-theoretic pricing of convertible bonds. Constantinides' results do no longer hold if the firm has issued additional debt as pointed out by Bühler/Koziol (2002). They also demonstrated that for this more complex capital structure the endogenous values of convertible bonds and stocks exhibit some strange properties in the block conversion case. By this reason and to avoid that the effects proved in Section 4 result from a non-equilibrium strategy, we consider the more sophisticated unrestricted conversion.

2.2 Values of the Convertible Bonds, Stocks, and the Additional Debt

(1) Maturity Date $t = T$

At maturity T and after redemption of the non-converted convertible bonds, the capital structure, conditional on no default, consists of $n + k_0 + k_T$ stocks and the additional debt only. The value of each individual stock $S_T\left(k_T, k_0, V_T^-\right)$ and the total debt value $D_T\left(k_T, k_0, V_T^-\right)$ of additional bonds follows from Merton (1974):

$$S_T\left(k_T, k_0, V_T^-\right) = \frac{1}{n + k_0 + k_T} C_T\left(V_T\right), \tag{4}$$

$$D_T\left(k_T, k_0, V_T^-\right) = FD \cdot e^{-r\cdot(TD-T)} - P_T\left(V_T\right) \tag{5}$$

Here $C_T\left(V_T\right)$ and $P_T\left(V_T\right)$ denote the value of a European call and a European put on the firm value V_T after redemption of all convertible bonds. The expiration dates of both options coincide with the maturity date of the additional debt and their strike prices with its the face value.

To analyze the value of a convertible bond at time T, we have to assume $k_0 < m$. $W_T\left(k_T, k_0, V_T^-\right)$ in T can possibly deviate from the value $W_T^+\left(k_T, k_0, V_T^-\right)$ if the bond is not converted. If the conversion volume public information, the stock value is continuous over time and $W_T\left(k_T, k_0, V_T^-\right)$ reads

$$W_T\left(k_T, k_0, V_T^-\right) = \begin{cases} \frac{(m-k_0-k_T)\cdot F + k_T \cdot S_T\left(k_T, k_0, V_T^-\right)}{m-k_0}, & \text{if } V_T^- \geq (m-k_0-k_T)\cdot F, \\ \frac{V_T^-}{m-k_0-k_T}, & \text{otherwise.} \end{cases} \tag{6}$$

$$= \frac{V_T^- - (n+k_0)\cdot S_T\left(k_T, k_0, V_T^-\right) - D_T\left(k_T, k_0, V_T^-\right).}{m-k_0} \tag{7}$$

Therefore, the value of a convertible bond before conversion can be either represented by the convex combination (6) of a claim on the face value F and the conversion value $S_T\left(k_T, k_0, V_T^-\right)$ or as the residual value of the firm value before redemption V_T^- minus the total value of stocks and additional debt as in (7).

The value $W_T^+\left(k_T, k_0, V_T^-\right)$ of a non-converted convertible bond at maturity is equal to the redemption value F when the firm value before redemption suffices for the redemption payment, or it equals the pro rata liquidation payment:

$$W_T^+\left(k_T, k_0, V_T^-\right) = \begin{cases} F, & \text{if } V_T^- \geq (m-k_0-k_T)\cdot F, \\ \frac{V_T^-}{m-k_0-k_T}, & \text{otherwise.} \end{cases}$$

In the case of a complete conversion, $W_T^+\left(m-k_0, k_0, V_T^-\right)$ equals F and can be understood as the value that one individual investor obtains for his non-converted bonds given that all the other bondholders convert their position completely.

(2) Call Date $t = 0$

At the call date $t = 0$, the asset values also depend on the call decision i_C of the firm. In the case of a call, $i_C = 1$, the convertible bond holders still have the possibility to convert some or all of their bonds. The non-converted convertible bonds are redeemed at the call price C. After the conversion decisions only stocks and additional bonds are outstanding. Their values can be determined by means of the same European options as at the maturity T of the convertible bonds:

$$S_0\left(1, k_0, V_0^-\right) := S_0\left(i_C = 1, k_0, V_0^-\right) = \frac{1}{n+k_0} C_0\left(V_0\right), \tag{8}$$

$$D_0\left(1, k_0, V_0^-\right) := D_0\left(i_C = 1, k_0, V_0^-\right) = FD \cdot e^{-r \cdot TD} - P_0\left(V_0\right), \tag{9}$$

Again, the value of a convertible bond $W_0\left(i_C=1,k_0,V_0^-\right)$ before conversion can be represented as a convex combination of the call price C and the stock price as in (10) or as a residual value as in (11):

$$\begin{aligned} W_0\left(1,k_0,V_0^-\right) &:= W_0\left(i_C=1,k_0,V_0^-\right) \\ &= \begin{cases} \frac{(m-k_0)\cdot C+k_0\cdot S_0\left(1,k_0,V_0^-\right)}{m}, & \text{if } V_0^- \geq (m-k_0)\cdot C, \\ \frac{V_0^-}{m-k_0}, & \text{otherwise.} \end{cases} \qquad (10) \\ &= \frac{V_0^- - n\cdot S_0\left(1,k_0,V_0^-\right) - D_0\left(1,k_0,V_0^-\right)}{m} \qquad (11) \end{aligned}$$

When a convertible bond is not converted after a call, its value $W_0^+\left(i_C=1,k_0,V_0^-\right)$ equals the call price C or the liquidation payment if the firm defaults:

$$W_0^+\left(1,k_0,V_0^-\right) := W_0^+\left(i_C=1,k_0,V_0^-\right) = \begin{cases} C, & \text{if } V_0^- \geq (m-k_0)\cdot C, \\ \frac{V_0^-}{m-k_0}, & \text{otherwise.} \end{cases}$$

In the opposite case that the firm does not call the convertible bond, $i_C=0$, the values of a non-converted convertible bond, $W_0^+\left(0,k_0,V_0^-\right)$, a stock, $S_0\left(0,k_0,V_0^-\right)$, and the additional debt, $D_0\left(0,k_0,V_0^-\right)$, are given by discounting their expected values at maturity

$$W_0^+\left(0,k_0,V_0^-\right) = e^{-r\cdot T}\int_0^\infty W_T\left(k_T^*,k_0,V_T^-\right)\cdot f\left(V_T^-|V_0\right)dV_T^-, \qquad (12)$$

$$S_0\left(0,k_0,V_0^-\right) = e^{-r\cdot T}\int_0^\infty S_T\left(k_T^*,k_0,V_T^-\right)\cdot f\left(V_T^-|V_0\right)dV_T^-, \qquad (13)$$

$$D_0\left(0,k_0,V_0^-\right) = e^{-r\cdot T}\int_0^\infty D_T\left(k_T^*,k_0,V_T^-\right)\cdot f\left(V_T^-|V_0\right)dV_T^-. \qquad (14)$$

Here $f\left(V_T^-|V_0\right)$ denotes the risk-neutral density function of the firm value for a given firm value V_0 at time $t=0$. As at maturity, we define $W_0^+\left(0,m,V_0^-\right)$ by $\lim_{x\to m} W_0^+\left(0,x,V_0^-\right)$. Intuitively, this is the value of an individual non-converted convertible bond if all the other bonds are converted. The asset values (12) to (14) are conditional to an exogenous conversion volume k_0 and an optimal conversion volume k_T^* at the maturity T. The optimal conversion volume k_T^* will be analyzed in Section 3.

The value of a convertible bond before conversion, $W_0\left(i_C=0,k_0,V_0^-\right)$, follows from the identity that all outstanding claims of the firm sum to V_0^- and is given by

$$W_0\left(0,k_0,V_0^-\right) = \frac{V_0^- - n\cdot S_0\left(0,k_0,V_0^-\right) - D_0\left(0,k_0,V_0^-\right)}{m}.$$

3 Optimal Conversion Strategy

3.1 Analysis at Maturity of the Convertible Bond

To analyze the optimal call and conversion strategies in $t = 0$ together with the value of the convertible bonds we proceed by a typical backward procedure. In the first step, we discuss the optimal conversion strategy k_T^* at the maturity T of the convertible bonds under the conditions that these bonds were not called and k_0 $(0 \leq k_0 < m)$ bonds were converted in $t = 0$.

As discussed in Section 2.1, each individual convertible bond holder acts under perfect competition and his conversion decision has no price impact. Following Constantinides/Rosenthal (1984), Schmeidler (1973), and Blonski (1999), we model this conversion competition by a non-atomic game. Using their methods, we can show that there exists at least one Nash equilibrium in the aggregate conversion volume k_T^*. In the following, we will exploit the special structure of the conversion game to characterize the properties of k_T^*.

Result 1

(1) If the firm defaults conditional on no conversion at T, i.e. $(m - k_0) \cdot F > V_T^- > 0$, then $k_T^* = 0$ holds.

(2) There exist two unique critical firm values $\underline{V}_T(k_0)$, $\overline{V}_T$ $\left(\underline{V}_T(k_0) < \overline{V}_T\right)$ such that

$$k_T^* = 0, \text{ if } V_T^- \leq \underline{V}_T(k_0),$$
$$k_T^* = m - k_0, \text{ if } V_T^- \geq \overline{V}_T.$$

Result 1 and the following Result 2 are from Bühler/Koziol (2002). We will present the proof for completeness and in order to give an economic intuition about the structure of the Nash equilibrium k_T^*.

Due to the missing individual price impact, every convertible bond holder compares the value of a non-converted convertible bond, $W_T^+\left(k_T, k_0, V_T^-\right)$, with the conversion value, $S_T\left(k_T, k_0, V_T^-\right)$. He will fully convert his position if $S_T\left(k_T, k_0, V_T^-\right) > W_T^+\left(k_T, k_0, V_T^-\right)$ holds, and he will not convert any of his bonds if S_T is below W_T^+. If $0 < V_T^- \leq (m - k_0) \cdot F$, i.e. if the firm defaults conditional on $k_T = 0$, then for an arbitrary conversion volume k_T $(0 \leq k_T \leq m - k_0)$, the value of a non-converted convertible bond

$$W_T^+\left(k_T, k_0, V_T^-\right) = \min\left\{\frac{V_T^-}{m - k_0 - k_T}, F\right\}$$

is higher than the upper bound (15) on the stock value:

$$S_T\left(k_T, k_0, V_T^-\right) \leq \frac{V_T^- - (m - k_0 - k_T) \cdot \min\left\{\frac{V_T^-}{m-k_0-k_T}, F\right\}}{n + k_0 + k_T} \tag{15}$$

This upper bound in relation (15) is the value of a stock if the firm has no additional debt. As a consequence, no convertible bond holder will convert and $k_T^* = 0$ is the unique Nash equilibrium for $0 < V_T^- \leq (m - k_0) \cdot F$.

The first critical firm value $\underline{V}_T(k_0)$ is defined by the condition[3]

$$W_T(k_T = 0, k_0, \underline{V}_T) = S_T(k_T = 0, k_0, \underline{V}_T) = F = W_T^+(k_T = 0, k_0, \underline{V}_T). \tag{16}$$

If (16) holds, each convertible bond holder is indifferent between a conversion and non-conversion. As the stock price strictly increases in the firm value if the firm does not default no convertible bond holder converts his position if $(m - k_0) \cdot F < V_T^- < \underline{V}_T$. This result together with Proposition (1) implies that $k_T^* = 0$ is the unique Nash equilibrium for $0 < V_T^- < \underline{V}_T$, and that the critical firm value $\underline{V}_T$ is unique.

The second critical firm value $\overline{V}_T$ is also defined by an indifference relation. This relation differs from (16) that it is conditional on a complete conversion:

$$\begin{aligned} W_T\left(k_T = m - k_0, k_0, \overline{V}_T\right) &= S_T\left(k_T = m - k_0, k_0, \overline{V}_T\right) \\ &= F = W_T^+\left(k_T = m - k_0, k_0, \overline{V}_T\right) \end{aligned} \tag{17}$$

Again, if follows from the strict monotonicity of the stock price in V_T^- that each convertible bond holder will convert his position completely whenever the stock price is above the face value F. This implies that $k_T^* = m - k_0$ is the unique Nash equilibrium for $V_T^- > \overline{V}_T$.

The relation $\underline{V}_T < \overline{V}_T$ follows from the representation (4) of $S_T\left(k_T, k_0, V_T^-\right)$, the indifference Equations (16) and (17), and the continuity of the stock price in t for an optimal conversion volume k_T^*. These properties imply $C_T(\underline{V}_T) = (n + k_0) \cdot F$ and $C_T\left(\overline{V}_T\right) = (n + m) \cdot F$, and therefore, $\underline{V}_T < \overline{V}_T$ by the strict monotonicity of the call value in V_T^-.

[3]To simplify the notation, we omit the dependency of $\underline{V}_T$ from k_0.

Result 2

(1) There exists a unique Nash equilibrium k_T^* in the aggregated volume k_T^* for each firm value V_T^- before redemption.

(2) For firm values V_T^- between $\underline{V}_T(k_0)$ and $\overline{V}_T$ the Nash equilibrium k_T^* is positive, continuous, and strictly increasing in V_T^-.

Result 2 especially states that at maturity a *partial conversion* is optimal if the firm value V_T^- lies between the two critical firm values.

Proposition (1) has already been shown for firm values $V_T^- \leq \underline{V}_T$ and $V_T^- \geq \overline{V}_T$. Now consider an arbitrary firm value $\underline{V}_T < V_T^- < \overline{V}_T$. We claim that there exists an aggregate conversion volume k_T^* $(0 < k_T^* < m - k_0)$ such that the indifference relation

$$S_T\left(k_T^*, k_0, V_T^-\right) = W_T^+\left(k_T^*, k_0, V_T^-\right) = F \tag{18}$$

holds. If (18) had no solution, either $S_T\left(k_T, k_0, V_T^-\right) > F$ or $S_T\left(k_T, k_0, V_T^-\right) < F$ would hold for all feasible k_T. In the first case, the monotonicity and continuity of S_T in V_T^- implies $S_T\left(m - k_0, k_0, V_T^-\right) > F$ in contradiction to the definition of $\overline{V}_T$. Analogously, we obtain in the second case a contradiction to the definition of $\underline{V}_T(k_0)$.

To prove uniqueness, we first show that the stock price has a negative slope at each k_T^* satisfying the indifference condition (18). Using the relation (4) and Equation (18), we obtain

$$\left.\frac{\partial S_T\left(k_T, k_0, V_T^-\right)}{\partial k_T}\right|_{k_T^*} = F \cdot \frac{C_T'\left(V_T^-(m - k_0 - k_T^*) \cdot F\right) - 1}{n + k_0 + k_T^*} < 0$$

as the call delta is smaller than one. The negative reaction of the stock price on an increase of k_T is caused by the wealth transfer from the stockholders to the holders of additional debt by reducing the probability of default in TD. If there existed more than one conversion volume that satisfies (18), for at least one volume k_T the slope would be positive. This contradicts the statement above.

k_T^* must be positive for $V_T^- > \underline{V}_T$ as $k_T^* = 0$ together with the strict monotonicity of the stock price in the firm value result in a contradiction of the indifference condition (18). The continuity of k_T^* in the firm value is an immediate consequence of the strict monotonicity of the stock price in V_T^-, the continuity of the stock price in k_T, and the implicit function theorem.

To verify the last part of proposition (2), the strict monotonicity of k_T^* in V_T^-, we increase V_T^- by $\Delta V > 0$. This change of V_T^- would also increase the stock price if

k_T^* did not change, such that the indifference condition (18) was violated. By an increase of k_T^* and the induced wealth transfer in favor of the additional bondholders, the stock price decreases until it equals again the face value F.

As a consequence of the optimal conversion strategy, the equilibrium values of a convertible bond before conversion, $W_T\left(k_T^*, k_0, V_T^-\right)$, a non-converted convertible bond, $W_T^+\left(k_T^*, k_0, V_T^-\right)$, and a stock, $S_T\left(k_T^*, k_0, V_T^-\right)$, are continuous and (weakly) increasing in the firm value. $W_T\left(k_T^*, k_0, V_T^-\right)$ is strictly increasing for firm values in the intervals $[0, (m-k_0)\cdot F]$ and $\left[\overline{V}_T, \infty\right)$. In the intervals $[0, \underline{V}_T)$, $W_T\left(k_T^*, k_0, V_T^-\right)$ is strictly larger than the stock value $S_T\left(k_T^*, k_0, V_T^-\right)$, and otherwise both values coincide. For firm values between the two critical firm values, both assets have the same value equal to the face value F of an individual convertible bond. In this interval of firm values an increase of V_T^- improves exclusively the position of the additional debt holders.

3.2 Analysis at the Call Date

The optimal conversion strategy k_T^* at maturity of the convertible bond allows us to compute from the relations (12) and (13) the values of a convertible bond $W_0^+\left(0, k_0, V_0^-\right)$ and a stock $S_0\left(0, k_0, V_0^-\right)$ before maturity.[4] These values are needed to determine the optimal conversion volume k_0^* and the optimal call decision i_C^* at the call date $t = 0$.

If the firm does not call ($i_C = 0$), the value of a non-converted convertible bond $W_0^+\left(0, k_0, V_0^-\right)$ always exceeds the stock value $S_0\left(0, k_0, V_0^-\right)$ and no conversion is the unique conversion strategy. This result is an immediate consequence of the assumption that the stock pays no dividends and the strict first order dominance of $S_T\left(k_T^*, k_0, V_T^-\right)$ by $W_T\left(k_T^*, k_0, V_T^-\right)$ as shown at the end of the previous section.

In the case of a call, a conversion can occur. The analysis of the optimal conversion strategy parallels that in Section 3.1. The only difference is that the individual convertible bond holder now compares the stock value $S_0\left(1, k_0, V_0^-\right)$ and the value $W_0^+\left(1, k_0, V_0^-\right)$ of the non-converted bond to make his optimal conversion decision.

The structure of the aggregate conversion volume k_0^* in equilibrium is again driven by two critical firm values $\underline{V}_0$ and $\overline{V}_0$ before redemption of the non-converted bonds. The first critical firm value is defined by

$$S_0\left(1, 0, \underline{V}_0\right) = W_0^+\left(1, 0, \underline{V}_0\right) = C. \tag{19}$$

[4]Note that in (12) and (13) k_T^* depends on V_T^-.

For the firm value $\underline{V}_0$ each convertible bond holder is indifferent between conversion and non-conversion if all the other convertible bonds are not converted ($k_0 = 0$). The indifference relation (19) corresponds to (16), where the face value F is replaced by the call price C.

The second critical firm value is defined analogously to (17) by the indifference relation conditional on $k_0 = m$, i.e. all the other bonds are fully converted:

$$S_0\left(1, m, \overline{V}_0\right) = W_0^+\left(1, m, \overline{V}_0\right) = C \tag{20}$$

Using these definitions, we can prove the following result:

Result 3

(1) The critical firm values $\underline{V}_0$ and $\overline{V}_0$, defined by (19) and (20), exist and are unique. For these critical values the relation

$$m \cdot C < \underline{V}_0 < \overline{V}_0$$

holds.

(2) Conditional on a call, the aggregate conversion strategy

$$k_0^* = \begin{cases} 0, & V_0^- < \underline{V}_0 \\ x \text{ with } W_0^+\left(1, x, V_0^-\right) = C = S_0\left(1, x, V_0^-\right), & \underline{V}_0 \leq V_0^- \leq \overline{V}_0 \\ m, & V_0^- > \overline{V}_0 \end{cases}$$

is the unique Nash equilibrium.

(3) For firm values between $\underline{V}_0$ and $\overline{V}_0$, the Nash equilibrium k_0^* is positive, continuous, and strictly increasing in V_0^-.

The values of a convertible bond $W_0\left(0, k_0^*, V_0^-\right)$, a non-converted convertible bond $W_0^+\left(0, k_0^*, V_0^-\right)$, and a stock $S_0\left(0, k_0^*, V_0^-\right)$ given that no call is announced are continuous and strictly increasing in the firm value V_0^-. Conditional on a call, these asset values are also continuous but only weakly increasing in V_0^-. The difference compared to the case that the bonds are not called is that for all firm values $V_0^- \in \left[\underline{V}_0, \overline{V}_0\right]$ the value of these assets equals the call price C. These properties will be used in Section 4.3. There, a graphical representation of these values can be found conditional on the optimal call policy.

4 Optimal Call Policy

4.1 Analysis of the optimal Call Strategy

By assumption the firm acts in the interest of the old stockholders. Therefore, it will call the outstanding convertible bonds if and only if the stock price strictly increases, conditional on the subsequent optimal conversion decision of the convertible bond holders. As the firm is able to anticipate $k_0^*\left(V_0^-\right)$ and $k_T^*\left(V_T^-\right)$, it can base its call decision on the sign of the stock value difference

$$\Delta S := S_0\left(1, k_0^*, V_0^-\right) - S_0\left(0, 0, V_0^-\right). \tag{21}$$

In (21) we made use of the fact that k_0^* is the optimal aggregated conversion volume for $i_C = 0$, i.e. if the bonds are not called. Whenever ΔS is positive, the firm will call and vice versa. To an analysis of an uninteresting case, we assume that for $\Delta S = 0$ the firm will not call.

Using Result 3 on the optimal aggregate conversion volume k_0^*, Figure 2 represents the stock value $S_0\left(1, k_0^*, V_0^-\right)$ conditional on a call for four intervals of the firm value V_0^-. $S_0\left(1, k_0^*, V_0^-\right)$ is zero if the firm value is below the redemption amount $m \cdot C$.For firm values $m \cdot C \leq V_0^- < \underline{V}_0$, the value of a stock follows from (8), taking into account that it is optimal not to convert ($k_0^* = 0$), and the relation $V_0 = V_0^- - m \cdot C$ between the firm value after and before repayment of the convertible bonds. C_0 is the value of the firm's total equity. If the firm value is between the two critical firm values, the Nash equilibrium is characterized by a partial conversion and the indifference relation $S_0\left(1, k_0^*, V_0^-\right) = C$. For firm values $V_0^- \geq \overline{V}_0$, a full conversion $k_0^* = m$ is optimal and stock values follow again from (8) and the relation $V_0 = V_0^-$.

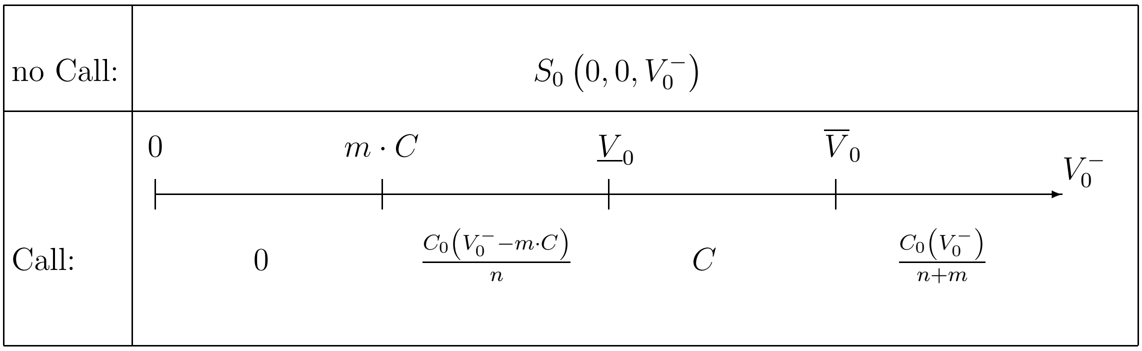
Consider a fixed firm value V_0^-. As the aggregated values of the three outstanding types of securities must sum up to the firm value

$$V_0^- = n \cdot S_0\left(i_C, k_0, V_0^-\right) + m \cdot W_0\left(i_C, k_0, V_0^-\right) + D_0\left(i_C, k_0, V_0^-\right)$$

the change ΔS of the stock value caused by the call decision can be decomposed into a change of the convertible bond value ΔW and a change of the debt value ΔD:

$$\Delta S = \frac{1}{n} \cdot \left(-m \cdot \Delta W - \Delta D\right), \tag{22}$$

Figure 2: Stock Values with and without a Call



where

$$\Delta W = \frac{(m-k_0^*)\cdot W_0^+\left(1,k_0^*,V_0^-\right) + k_0^* \cdot S_0\left(1,k_0^*,V_0^-\right)}{m} - W_0^+\left(0,0,V_0^-\right), \qquad (23)$$

$$\Delta D = D_0\left(1,k_0^*,V_0^-\right) - D_0\left(0,0,V_0^-\right) \qquad (24)$$

In the case that the firm has not issued additional debt, ΔD equals zero and the IBS policy to call a convertible bond whenever the non-called value $W_0^+\left(0,0,V_0^-\right)$ is above C, ($\Delta W < 0$), can be easily derived from (22) and (23). If $W_0^+\left(0,0,V_0^-\right)$ exceeds C, the first term in the difference (23) of ΔW equals $\max\left(C, S_0\left(1,m,V_0^-\right)\right)$ and is lower than $W_0^+\left(0,0,V_0^-\right)$. The reason for $W_0^+\left(0,0,V_0^-\right) > S_0\left(1,m,V_0^-\right)$ is the first order dominance of the stock value at maturity by the convertible bond value which we have already used before. In the opposite case $W_0^+\left(0,0,V_0^-\right) < C$, a call is not optimal ($\Delta W > 0$) because it either results in a default of the firm or the first term in the difference (23) equals C.

As formula (22) indicates, the IBS strategy need not to be optimal if firms have additional debt outstanding. For example, we will show that the call induced wealth transfer $m\cdot\Delta W$ from the convertible bond holders to the stockholders can be positive ($\Delta W < 0$) and, simultaneously, the wealth transfer ΔD from the stockholders to the additional bondholders is also positive ($\Delta D < 0$) and larger. We will also show that the converse is possible, too. As the wealth transfer to the additional bondholders ΔD caused by a call is of special importance, we analyze this effect first.

4.2 Analysis of the Effect of a Call on the Debt Value

In the following sections, we assume that the call price C is equal to the face value F of an individual convertible bond. The analysis of the cases $C > F$ and $C < F$

parallels that for $C = F$.

In the subsequent discussion, we consider two cases. In the first one, we compare the debt value $D_0\left(1, m, V_0^-\right)$ under an enforced full conversion with the debt value $D_0\left(1, k_0, V_0^-\right)$ under a partial conversion $0 \leq k_0 < m$, and with the debt value $D_0\left(0, 0, V_0^-\right)$ if the bonds are not called. The second, more difficult case compares the two debt values $D_0\left(1, 0, V_0^-\right)$ and $D_0\left(0, 0, V_0^-\right)$ if the conversion volume is zero independent whether the convertible bonds are called or not.

The value $D_0\left(i_C, k_0, V_0^-\right)$ is basically determined by the distribution of the firm value V_{TD}^- before redemption at the maturity of the additional bonds. We use the distribution that results from an enforced full conversion in $t = 0$ and the corresponding debt value $D_0\left(1, m, V_0^-\right)$ as reference case. If the convertible bonds are only partially converted in $t = 0$, the firm value at $t = 0$ is reduced by $(m - k_0) \cdot F$ and therefore the distribution of V_{TD}^- shifts to the left. Hence, the firm value V_{TD}^- and the debt value are strictly first order stochastically dominated by those in the reference case. The same result holds if the bonds are not called (and therefore not converted) because the redemption volume $(m - k_T^*)$ at the maturity date T is positive as long as $V_T^- < \overline{V}_T$. These strict dominance results imply

$$D_0\left(1, m, V_0^-\right) > D_0\left(0, k_0, V_0^-\right) \tag{25}$$

and

$$D_0\left(1, m, V_0^-\right) > D_0\left(0, 0, V_0^-\right). \tag{26}$$

The relations (25) and (26) indicate that a forced complete conversion results in a wealth transfer to the debt holders $\Delta D > 0$. This result completes the analysis of the first case.

In the second case, we compare the debt value $D_0\left(1, 0, V_0^-\right)$ in case of a call and a conversion volume $k_0 = 0$ with the debt value $D_0\left(0, 0, V_0^-\right)$ in the absence of a call. The difference $D_0\left(1, 0, V_0^-\right) - D_0\left(0, 0, V_0^-\right)$ depends on the firm value V_0^- and is driven by four effects:

(1) If the firm value V_0^- is below $m \cdot F$, a call of the convertible bond causes a default and $D_0\left(1, 0, V_0^-\right)$ equals zero. Therefore, $D_0\left(1, 0, V_0^-\right) - D_0\left(0, 0, V_0^-\right)$ is negative.

(2) If the firm value V_0^- is sufficiently large and the bonds are not called, then the probability that they will be fully converted $(k_T^* = m)$ at maturity is close to one. If the bonds are called, the firm value drops by $m \cdot F$ and the distribution of V_{TD}^- at maturity of the additional debt will be strictly first order stochastically dominated by the distribution in the no call case. This implies that for sufficiently large firm values V_0^-, $D_0\left(1, 0, V_0^-\right) - D_0\left(0, 0, V_0^-\right)$ is also negative. As for increasing

V_0^- the default probability of the additional debt converges to zero, and the value of debt converges to the discounted face value FD independent of the call decision. Therefore, the difference $D_0\left(1,0,V_0^-\right) - D_0\left(0,0,V_0^-\right)$ also converges to zero.

(3) For medium firm values two contrary effects have an impact on the distribution of V_{TD}^- and, consequently, on the difference $D_0\left(1,0,V_0^-\right) - D_0\left(0,0,V_0^-\right)$.

(a) *Average effect*: The expected value of V_{TD}^- under the risk-neutral measure is higher if no call occurs compared with the case of a call. This is due to the fact that the reduction $m \cdot F$ in the firm value is certain if the bond is called, whereas the reduction $(m - k_T^*) \cdot F$ is uncertain and possibly smaller in the no call case. In addition, as the firm value has a positive drift, the forgone return on $m \cdot F$ until T reduces the expected value further if the bonds are called. Therefore, a call has a negative effect on the difference $D_0\left(1,0,V_0^-\right) - D_0\left(0,0,V_0^-\right)$.

(b) *Spread effect*: We consider two rates of return $y > 0$ and $-y$ of the firm value process (without redemption payments) for the period from $t = 0$ to $t = T$. We assume $V_0^- > m \cdot F$ and $m \cdot F < V_0^- \cdot e^{-y} < V_0^- \cdot e^{y} < \underline{V}_T$, i.e. no default occurs at $t = 0$ and $t = T$, and at T it is optimal not to convert the convertible bonds. Then, the following inequalities hold:

$$V_0^- \cdot e^{-y} - m \cdot F < \left(V_0^- - m \cdot F\right) \cdot e^{-y} < \left(V_0^- - m \cdot F\right) \cdot e^{y} < V_0^- \cdot e^{y} - m \cdot F \quad (27)$$

In other words, for two symmetric rates of return y and $-y$ the corresponding spread of firm values at maturity of the convertible bonds is larger if the bonds are not called as if they are called. This result holds under the stated condition that it is optimal not to convert the bonds at $t = T$.

If the inequalities (27) are multiplied by an arbitrary compound factor $e^{\overline{y}}$ for the period from T to TD, the spread relation also holds for the maturity date of the additional debt. As the intrinsic value of the additional debt is concave, the debt value benefits from a lower spread. Therefore, a call also has a positive effect on the difference $D_0\left(1,0,V_0^-\right) - D_0\left(0,0,V_0^-\right)$. The aggregation of the two effects (a) and (b) determine the sign of this difference for medium firm values.

We summarize our discussion in

Result 4

(1) A call together with a full conversion implies a wealth transfer to the additional debt holders relative to a call with a partial conversion $(0 \leq k_0 < m)$ *or relative to the decision not to call.*

(2) If the firm value V_0^- is below the outstanding convertible bond volume $m \cdot F$, or if the firm value is sufficiently high, a call together with no conversion ($k_0 = 0$) has a negative wealth effect for the additional debt holders relative to the case not to call.

(3) For medium firm values V_0^- there are two contrary effects of a call together with no conversion ($k_0 = 0$) on the wealth of the convertible debt holders. They can result in a positive or negative aggregate wealth effect.

Figure 3: Debt Value Change from a Call without Conversions

The diagram shows the difference $D_0\left(1,0,V_0^-\right) - D_0\left(0,0,V_0^-\right)$ as a function of the firm value V_0^-. The parameter values are $n = 100$, $m = 100$, $C = F = 100$, $FD = 100{,}000$, $TD = 2$, $T = 1$, $r = 0.05$, and $\sigma = 0.5$.

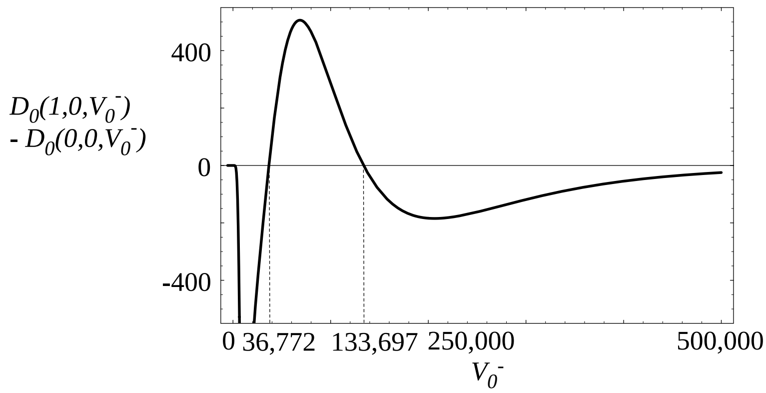
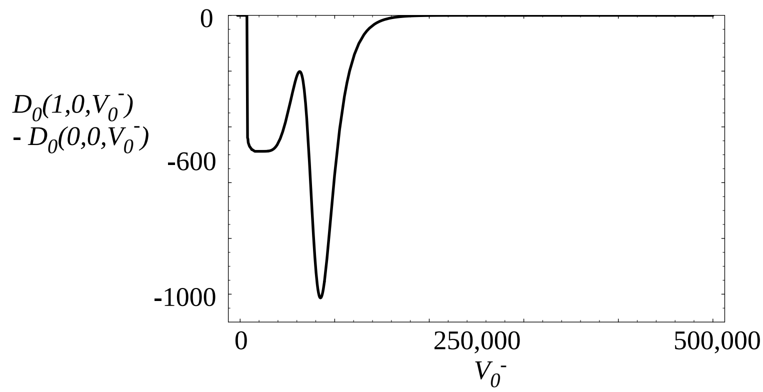


Figure 3 shows a first example for propositions (2) and (3) of Result 4. For low firm values $V_0^- < m \cdot F = 10{,}000$ a call triggers a default and reduces the wealth of the debt holders, i.e. $D_0\left(1,0,V_0^-\right) - D_0\left(0,0,V_0^-\right) < 0$. If the firm value increases the spread effect becomes more important relative to the average effect and a call results in positive wealth consequences for the debt holders for firm values between $36,772$ and $133,697$ if in addition $k_0^* = 0$ holds. For sufficiently high firm values, $D_0\left(0,0,V_0^-\right)$ exceeds $D_0\left(1,0,V_0^-\right)$ as a complete conversion at maturity, that affects $D_0\left(0,0,V_0^-\right)$ positively, has a probability close to one. This shape of $D_0\left(1,0,V_0^-\right) - D_0\left(0,0,V_0^-\right)$ with a local minimum followed by one local maximum and a further local minimum is typical for all examples considered.

However, as Figure 4 indicates the local maximum does not need to be positive and a call with no conversion is always nonbeneficial for the debt holders compared to to the case not to call. The second example differs from the first one by a lower volatility σ of the firm's return and by the longer maturity TD of the additional debt. The lower volatility reduces the spread effect and the longer maturity increases the average effect.

Figure 4: Debt Value Change from a Call without Conversions

The diagram shows the difference $D_0\left(1,0,V_0^-\right) - D_0\left(0,0,V_0^-\right)$ *as a function of the firm value* V_0^-. *The parameter values are* $n = 100$, $m = 100$, $C = F = 100$, $FD = 100{,}000$, $TD = 10$, $T = 1$, $r = 0.05$, *and* $\sigma = 0.1$.



4.3 Properties of the Optimal Call Strategy

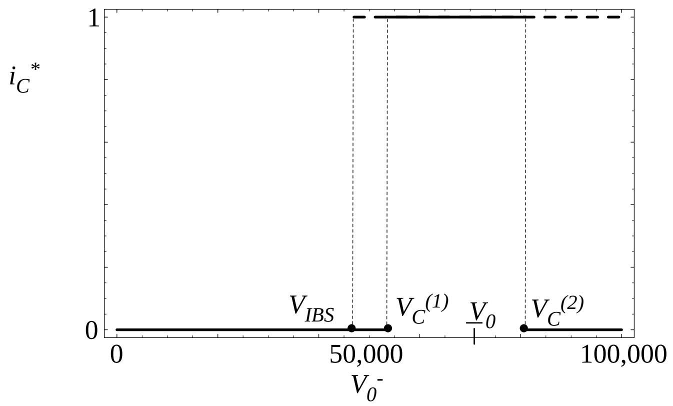
Under the IBS strategy, there is exactly one critical firm value V_{IBS} which separates the non-call region of firm values from the call region. This result follows from the continuity and strict positive monotonicity of $W_0^+\left(0,0,V_0^-\right)$ in V_0^- and from $W_0^+\left(0,0,V_0^-\right) < F$ ($> F$) for sufficiently small (large) firm values. If the firm has additional debt outstanding, the call policy differs substantially from the IBS strategy. We first present two examples to give an intuition for the typical structure of call policies.

Figure 5 shows the optimal IBS strategy (dashed line) and the optimal call policy.[5] The first important difference is that the firm optimally begins to call for a critical firm value $V_C^{(1)} = 53{,}138$ which is about 13 percent higher than under the IBS strategy. We denote this case as a *late call* although this notation is not precise as we do not consider the time dimension. It is correct if we assume a path of firm values that increase with time.

[5] Note that we do not compare the call strategies of two firms with different capital structures. We consider the same firm with additional debt that follows two different call strategies, the (non optimal) IBS strategy and the optimal call strategy.

Figure 5: Optimal Call Policy

The diagram shows the optimal call policy i_C^* as a function of the firm value V_0^-. The dashed line indicates the IBS strategy. The parameter values are $n = 100$, $m = 100$, $C = F = 100$, $FD = 100,000$, $TD = 2$, $T = 1$, $r = 0.05$, and $\sigma = 0.5$. The critical firm values are given by $V_{IBS} = 46,911$, $V_C^{(1)} = 53,138$, $\underline{V}_0 = 72,914$, $V_C^{(2)} = 81,075$, and $\overline{V}_0 = 82,324$.



The reason of the late call can be explained by means of Figure 3 that uses the same parameter values as Figure 5. As $\Delta W < 0$ holds for firm values $V_0^- > V_{IBS}$, ΔD must be positive on the interval $\left(V_{IBS}, V_C^{(1)}\right]$. A comparison with Figure 3 shows that, in fact, this interval is a subinterval of the interval $[36,772; 133,697]$.[6]

We summarize this discussion in

Result 5

A late call is possible only if at the indifference firm value V_{IBS} of the IBS strategy a call improves the wealth of the additional bondholders ($\Delta D > 0$).

Note that we did not prove that it is optimal to call the convertible bonds for any firm value.

Calling the convertible bonds for firm values $V_0^- < V_{IBS}$ can also be optimal. We denote this case as an *early call*. A sufficient condition is that $D_0\left(1, 0, V_0^-\right) - D_0\left(0, 0, V_0^-\right)$ is always negative as in Figure 4. To prove this assertion consider the critical firm value $\underline{V}_0$, the largest value for which it is optimal not to convert after a call. As the firm value V_0^- before redemption is independent of the call decision,

[6] As $V_C^{(1)}$ is smaller than $\underline{V}_0$ it is optimal not to convert and the difference in Figure 3 equals ΔD for firm values $V_0^- \leq \underline{V}_0$.

the following two equations hold for $V_0^- = \underline{V}_0$:

$$n \cdot S_0\left(1,0,\underline{V}_0\right) + m \cdot W_0^+\left(1,0,\underline{V}_0\right) = \underline{V}_0 - D_0\left(1,0,\underline{V}_0\right)$$
$$n \cdot S_0\left(0,0,\underline{V}_0\right) + m \cdot W_0^+\left(0,0,\underline{V}_0\right) = \underline{V}_0 - D_0\left(0,0,\underline{V}_0\right)$$

Since for $V_0^- = \underline{V}_0$ the stock value $S_0\left(1,0,\underline{V}_0\right)$ equals $W_0^+\left(1,0,\underline{V}_0\right)$, we obtain $S_0\left(1,0,\underline{V}_0\right) = \frac{\underline{V}_0 - D_0(1,0,\underline{V}_0)}{n+m}$ from the first equation. If the bonds are not called, $W_0^+\left(0,0,\underline{V}_0\right)$ is larger than $S_0\left(0,0,\underline{V}_0\right)$ and $S_0\left(0,0,\underline{V}_0\right) < \frac{\underline{V}_0 - D_0(0,0,\underline{V}_0)}{n+m}$ follows from the second equation. Therefore, the stock value after a call is larger than the stock value without a call:

$$S_0\left(1,0,\underline{V}_0\right) = \frac{\underline{V}_0 - D_0\left(1,0,\underline{V}_0\right)}{n+m} > \frac{\underline{V}_0 - D_0\left(0,0,\underline{V}_0\right)}{n+m} > S_0\left(0,0,\underline{V}_0\right)$$

This property follows from the condition $D_0\left(1,0,\underline{V}_0\right) - D_0\left(1,0,\underline{V}_0\right) < 0$. Thus, it is optimal to call the convertible bonds at the firm value $\underline{V}_0$. This implies that $V_C^{(1)}$ is smaller than $\underline{V}_0$ and that $V_{IBS} > V_C^{(1)}$ holds. The second assertion follows from the definition of V_{IBS} $\left(W_0^+\left(0,0,\underline{V}_0\right) = C \text{ which is equivalent to } \Delta W = 0\right)$, the assumption $\Delta D\left(\underline{V}_0\right) < 0$ and the definition of $V_C^{(1)}$ as the smallest firm value for which $\Delta S = 0$ holds.

Summing up, we obtain

Result 6

If $D_0\left(1,0,V_0^-\right) - D_0\left(0,0,V_0^-\right)$ *is negative for all firm values, then*

(1) $V_C^{(1)}$ *exists and*

(2) an early call is optimal, i.e. $V_C^{(1)} < V_{IBS}$ *holds.*

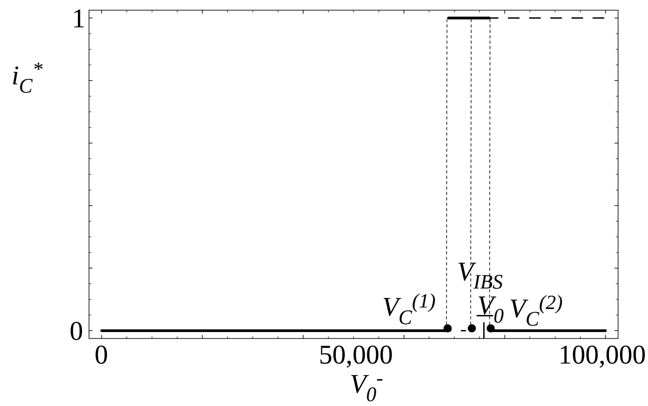
The intuition of Result 6 is easy to explain. As in the relevant call region it is optimal not to convert, a call always reduces the wealth of the additional bondholders. Therefore, it is optimal to call at firm values for which the call price is larger than the value of the non-converted convertible bond ($\Delta W < 0$).

Figure 6 presents an example for an early call. The parameter values underlying this figure are the same as in Figure 4. The critical firm value $V_C^{(1)} = 68,753$ is about 5% below the $V_{IBS} = 72,255$.

The second important, and possibly more surprising difference between the optimal and the IBS strategy concerns the structure of i_C^* as a function of the firm value. According to Figures 5 and 6 it is not optimal to call for sufficiently high firm values. The possible existence of a second critical firm value $V_C^{(2)}$ above which the firm does not call presents a fundamental difference to the IBS strategy.

Figure 6: Optimal Call Policy

The diagram shows the optimal call policy i_C^* *as a function of the firm value* V_0^-*. The dashed line indicates the IBS strategy. The parameter values are* $n = 100$*,* $m = 100$*,* $C = F = 100$*,* $FD = 100,000$*,* $TD = 10$*,* $T = 1$*,* $r = 0.05$*, and* $\sigma = 0.1$*. The critical firm values are given by* $V_C^{(1)} = 68,753$*,* $V_{IBS} = 72,255$*,* $\underline{V}_0 = 74,606$*,* $V_C^{(2)} = 76,948$*, and* $\overline{V}_0 = 78,044$*.*



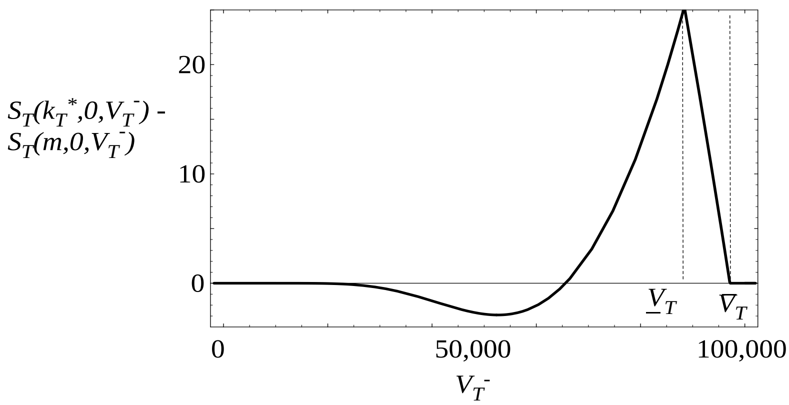
To give an intuition why it is optimal for the firm not to call the convertible bond issue for large firm values, we regard the consequences of a call and no call for a firm value $V_0^- > \overline{V}_0$. For this firm value a call forces a complete conversion, and the distribution of the firm value V_T^- at maturity before the conversion decision is the same whether the firm calls or does not call at $t = 0$. The two possible call decisions have the consequence that at maturity T of the convertible bonds either n stocks and m convertible bonds or $n + m$ stocks are outstanding.

Figure 7 shows the difference between the stock value $S_T\left(k_T^*, 0, V_T^-\right)$ at T under the optimal strategy k_T^* and the stock value $S_T\left(m, 0, V_T^-\right)$ after a complete conversion at $t = 0$. If the firm value V_T^- at maturity is low, the stockholders benefit from having forced the convertible bond holders to a complete conversion at $t = 0$ as, given the information on V_T^-, the convertible bond holders would decide not to convert. For firm values between $\underline{V}_T$ and $\overline{V}_T$, the stock value $S_T\left(k_T^*, 0, V_T^-\right)$ equals F, and the difference is positive because a complete conversion in $t = 0$ results in a larger wealth transfer to the debt holders at the cost of the stockholders than a partial conversion at T. This wealth transfer argument holds also for firm values that are below $\underline{V}_T$ but not too small. If the firm value is larger than $\overline{V}_T$, a complete conversion in T is optimal and therefore the difference equals zero.

The positive difference $S_T\left(k_T^*, 0, V_T^-\right) - S_T\left(0, m, V_T^-\right)$ of the stock values on the

Figure 7: Stock Value Differences at Maturity

The diagram shows the difference $S_T\left(k_T^*, 0, V_T^-\right) - S_T\left(m, 0, V_T^-\right)$ as a function of the firm value V_T^- conditional to the case that a call in $t = 0$ enforces a full conversion. The parameter values are $n = 100$, $m = 100$, $F = 100$, $FD = 100,000$, $TD - T = 1$, $r = 0.05$, and $\sigma = 0.5$. The critical firm values are given by $\underline{V}_T = 88,399$ and $\overline{V}_T = 97,136$.



interval $\left(\underline{V}_T, \overline{V}_T\right)$ allows for an interesting interpretation. If the convertible bond holders are not forced to a full conversion in $t = 0$ they later protect the stockholders against a too high wealth transfer by converting only partially. Without additional debt the stock value difference can never be strictly positive because the critical firm values $\underline{V}_T$ and $\overline{V}_T$ coincide and an ex-post suboptimal complete conversion at $t = 0$ is always beneficial for the stockholders.

This property of the stock value difference for firm values above $\underline{V}_T$ allows us to argue why a second critical firm value $V_C^{(2)}$ exists if and only if the firm has additional debt outstanding. We were not able to prove the existence and the uniqueness of $V_C^{(2)}$ for our continuous time model. But we are able to prove an analogous result for a binomial model.

A not necessarily recombining binomial tree is characterized by the initial firm value $\widehat{V}_0^-$, the length Δt of the time interval, the size of the up and down movements, and the risk-neutral probabilities. We determine these four terms such that the following conditions are satisfied. First, the tree is consistent with the given drift r and the instantaneous volatility σ. Second, $\widehat{V}_0^- \geq \overline{V}_0$, and third, a path that has only down movements hits exactly $\underline{V}_T$ at $t = T$.

As $\overline{V}_0 < \overline{V}_T$, it is easy to show that there exists at least one binomial tree that satisfies for an arbitrary but fixed Δt all four conditions. This tree has at least one final firm value in the interval $\left[\underline{V}_T, \overline{V}_T\right]$, and no firm values below $\underline{V}_T$. For this tree the stock value $S_0\left(0, 0, \widehat{V}_0^-\right)$ at the firm value $\widehat{V}_0^-$ is higher than the stock value $S_0\left(1, m, \widehat{V}_0^-\right)$ as the difference shown in Figure 7 is for at least one final firm value positive and for all others zero. Therefore, for firm value $V_0^- \geq \widehat{V}_0^-$ it is not optimal to call.[7]

For this discrete model we have

Result 7

For sufficiently high firm values it is not optimal to call.

Thus, in contrast to the IBS strategy it is not optimal to force a conversion by a call when it is possible. More precisely, it is optimal not to call and to prevent a conversion for high firm values.

4.4 Properties of the Values of a Convertible Bond and a Stock

In this section, we analyze the values of a convertible bond $W_0\left(i_C^*, k_0^*, V_0^-\right)$ and a stock $S_0\left(i_C^*, k_0^*, V_0^-\right)$ under the optimal call policy i_C^* and the optimal conversion decision k_0^* as a function of the firm value V_0^-. We assume in this section that $V_C^{(1)}$ and $V_C^{(2)}$ exist, that they are unique, and that $V_C^{(1)} < V_C^{(2)}$ holds.

(a) Convertible Bond Value

As reported at the end of section 3.2, the value $W_0\left(1, k_0^*, V_0^-\right)$ of the called and optimally converted bond as well as the value $W_0\left(0, 0, V_0^-\right)$ of the non-called convertible bond are continuous and strictly increasing in V_0^-. Therefore, these properties immediately transfer to the value $W_0^+\left(i_C^*, k_0^*, V_0^-\right)$ of an optimally called bond in the no-call regions $0 \leq V_0^- \leq V_C^{(1)}$ and $V_C^{(2)} \leq V_0^-$. The behavior of $W_0\left(1, k_0^*, V_0^-\right)$ in the call region depends on the location of $\overline{V}_0$ relative to V_C^2. If $\overline{V}_0 > V_C^{(2)}$ holds, the convertible bond value is constant over the call region as for firm values below $\overline{V}_0$. If $\overline{V}_0 < V_C^{(2)}$, then the convertible bond value strictly increases on the interval $\left(\overline{V}_0, V_C^{(2)}\right]$.

At the two critical firm values $V_C^{(1)}$ and $V_C^{(2)}$ that separate the call from the no-call region the convertible bond value typically exhibits jumps. The only exception is that the wealth of the additional debt holders is not affected by a change of i_C. As

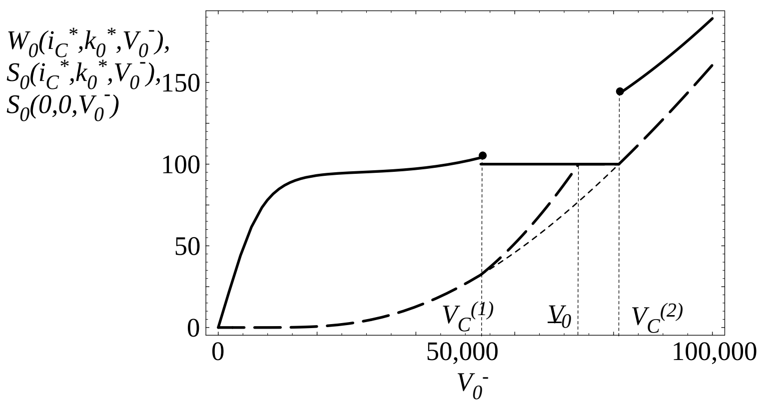
[7] Note that we defined $i_C^* = 0$ in the indifference case.

we will show below the stock price is continuous at these two critical firm values. Therefore, if a call increases the wealth of the additional debt holders, $\Delta D > 0$, the value of the convertible bonds jumps downwards, and vice versa.

Figure 8 gives an example of the convertible bond and the stock value as a function of the firm value. The parameters selected in Figure 8 are the same as in Figures 3 and 5. From the discussion of Figure 5 we know that there is a late call at $V_C^{(1)}$ and that the additional bondholders benefit from this call at the cost of the convertible bond holders. The value of the convertible bonds drops from 104.0 to the call price of 100.

Figure 8: Values of a Convertible Bond and a Stock

The diagram shows the values of a convertible bond $W_0\left(i_C^*, k_0^*, V_0^-\right)$ (solid line) and a stock $S_0\left(i_C^*, k_0^*, V_0^-\right)$ (dashed line) as a function of the firm value V_0^-. The stock value $S_0\left(0, 0, V_0^-\right)$ for firm values without a call is indicated by a dotted line. The parameter values are $n = 100$, $m = 100$, $C = F = 100$, $FD = 100,000$, $TD = 2$, $T = 1$, $r = 0.05$, and $\sigma = 0.5$. The critical firm values are given by $V_{IBS} = 46,911$, $V_C^{(1)} = 53,138$, $\underline{V}_0 = 72,914$, $V_C^{(2)} = 81,075$, and $\overline{V}_0 = 82,324$.

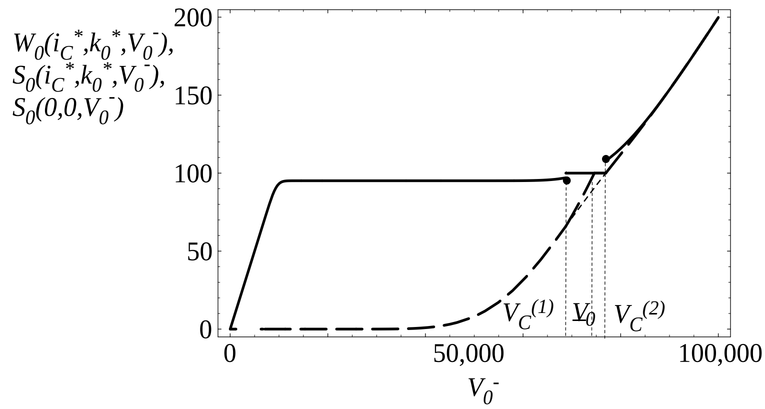


An example, in which the additional debt holders suffer from a call represents Figure 9. This figure corresponds to the previous Figures 4 and 6. As here the convertible bonds are called early relative to the IBS strategy, the convertible bond value jumps upwards at $V_C^{(1)}$ from 97.0 to the call price 100.

This example demonstrates that an issuer call right can add value to a convertible bond. It also shows that the traditional view of a *callable* convertible bond need not to be correct. This view starts from the insight that the convertible bond holders are short in a call. As the firm's call right has a positive value, a callable convertible bond should have a lower value than an otherwise identical but non-callable convertible

Figure 9: Values of a Convertible Bond and a Stock

The diagram shows the values of a convertible bond $W_0\left(i_C^*, k_0^*, V_0^-\right)$ *(solid line) and a stock* $S_0\left(i_C^*, k_0^*, V_0^-\right)$ *(dashed line) as a function of the firm value* V_0^-*. The stock value* $S_0\left(0,0,V_0^-\right)$ *for firm values without a call is indicated by a dotted line. The parameter values are* $n = 100$*,* $m = 100$*,* $C = F = 100$*,* $FD = 100,000$*,* $TD = 10$*,* $T = 1$*,* $r = 0.05$*, and* $\sigma = 0.1$*. The critical firm values are given by* $V_C^{(1)} = 68,753$*,* $V_{IBS} = 72,255$*,* $\underline{V}_0 = 74,606$*,* $V_C^{(2)} = 76,948$*, and* $\overline{V}_0 = 78,044$*.*



bond. This is correct if the firm has no additional debt outstanding. However, in the case of an early call, the firm can reduce the wealth of the debt holders in favor of the convertible bond holders (and stockholders). This wealth transfer explains why a call right can be advantageous for the convertible bond holders.

The second discontinuity of the convertible bond value at $V_C^{(2)}$ is associated with an upwards jump of the convertible bond value from its call price 100 to 143.0 in Figure 8 and 107.9 in Figure 9. If $V_C^{(2)}$ exceeds $\underline{V}_0$ as in these figures, the jump of $W_0\left(i_C^*, k_0^*, V_0^-\right)$ must have a positive sign. The rationale is that the convertible bond value $W_0\left(0,0,V_0^-\right)$ in case of no call exceeds the corresponding stock value $S_0\left(0,0,V_0^-\right)$. As $\underline{V}_0 < V_C^{(2)}$, for firm values $\underline{V}_0 < V_0^- < V_C^{(2)}$ a call enforces a partial or full conversion and $W_0\left(1, k_0^*, V_0^-\right) = S_0\left(1, k_0^*, V_0^-\right)$ holds. At $V_C^{(2)}$ the stock value is continuous, but the convertible bond value jumps upwards to $W_0\left(0,0,V_C^{(2)}\right)$ that is larger than $S_0\left(0,0,V_C^{(2)}\right) = S_0\left(1, k_0^*, V_C^{(2)}\right)$. These considerations provide us with the following result.

Result 8

Assume that there exist exactly two critical firm values at which the optimal call policy switches from $i_C^* = 0$ *to* $i_C^* = 1$ *and vice versa.*

(1) The value of a convertible bond is continuous and (weakly) increases with the firm value except for those firm values at which the optimal call policy changes. The convertible bond value exhibits a negative (positive) jump when the firm calls the convertible bond issue too late (early) relative to the IBS strategy. Conversely, the convertible bond value always has a positive jump at the second critical firm, if this critical firm value lies above $\underline{V}_0$.

(2) If the bond is called late, the call right increases the value of a convertible bond for some firm values.

(b) Stock Value

The stock value $S_0\left(i_C^*, k_0^*, V_0^-\right)$ is the maximum of the two continuous and increasing stock values $S_0\left(0, 0, V_0^-\right)$ and $S_0\left(1, k_0^*, V_0^-\right)$. Therefore, $S_0\left(i_C^*, k_0^*, V_0^-\right)$ is a (weakly) increasing function of the firm value. It is strictly increasing except for firm values for which a *partial* conversion is enforced by a call. It is also continuous, even at those firm values where the call policy changes. For firm values with an optimal partial conversion, the stock value equals the convertible bond value and is therefore equal to the call price F.

A further property is in conflict with the classical view of a callable convertible bond. The "old" stockholders have the option to call the convertible bonds at the strike price C. We expect from standard option pricing theory that a higher call price reduces the value of this call. As the value of this call right is part of the stock value, a higher call price should also reduce the stock value. But again, a call can force a conversion which induces wealth effects that can *increase* the stock value. This is the case, e.g. if a higher call price reduces the conversion volume and the lower conversion volume reduces the wealth transfer from the stockholders to the additional debt holders.

In the following, we will make this intuition precise. The critical firm values $\underline{V}_0$ and $V_C^{(2)}$ are continuous in the call price. Therefore, a small change of C will cause a small change of these two critical firm values and the intersection of the corresponding two intervals is not empty. For all firm values in this intersection, a partial conversion $0 < k_0^* < m$ is optimal for both call prices and the stock price is equal to the call price. Therefore, if C is increased by ΔC the stock values for these firm values also increase by ΔC.

To illustrate this effect, we consider the example presented in Figure 8. We know that for all firm values between $\underline{V}_0 = 72,914$ and $V_C^{(2)} = 81,075$ the stock value is equal to $C = 100$. If we increase the call price by $\Delta C = 10$ to 110, $\underline{V}_0$ increases to

$76,155$ and $V_C^{(2)}$ to $84,421$. For all firm values in the interval $(76,155; 81,075)$ the stock value increases also to 110.

We summarize our analysis of the stock value in

Result 9

(1) The value of a stock is continuous and (weakly) increases in the firm value. The monotonicity is strict whenever no partial conversion occurs.

(2) If it is optimal to convert partially the convertible bonds at the firm value V_0^-, a sufficiently small change of ΔC of the call price changes the stock value by the same amount.

5 Conclusion

This paper presents a new explanation for the empirical puzzle that callable convertible bonds are called too late relative to an optimal call strategy originally proposed by Ingersoll (1977a) and Brennan/Schwartz (1977) (IBS strategy). Contrary to the previous literature, we do not assume that there are frictions of any kind to explain the delay in the calls of convertible bonds. The only difference to the seminal papers by Ingersoll (1977a) and Brennan/Schwartz (1977) is that we assume a more general and more realistic capital structure. The firm considered has issued in addition to stocks and convertible bonds also subordinated straight bonds.

This seemingly innocuous generalization has the important consequence that a call can affect the convertible bond value differently as the value of the additional debt. In particular, a call causes a wealth transfer between the convertible bond, stock, and debt holders which has a remarkable effect on the call strategy. We made the following interesting findings:

(1) The widely accepted IBS strategy is no longer in the interest of the old stockholders, if firms have additional debt outstanding. Whenever a call improves the position of the debt holders, a call must be too late relative to the IBS strategy. Conversely, an early call occurs if a call hurts the debt holders. Figures 5 and 6 document that the deviations of the optimal strategy from the IBS strategy can be remarkable. As a result, the convertible bond value can be considerably larger than the call price and it is still not optimal for the firm to call. Conversely, a call can be optimal, though the convertible bond value is below the call price.

(2) As most firms have additional debt outstanding, we argue that the empirical

findings should not be compared with the non-optimal IBS strategy. Therefore, it is also questionable whether an empirical puzzle exists at all.

(3) If calling the convertible bonds is optimal, the conversion volume increases continuously. Contrary to the result by Ingersoll (1977a) and Brennan/Schwartz (1977), a call does not necessarily enforce a full conversion. On the other side, it is not always optimal to call if the call enforces a conversion.

(4) For high firm values, it is optimal for the firm not to call. The reason is that a forced full conversion before maturity induces a definite wealth transfer to the debt holders. If the convertible bonds are not called, the conversion is delayed until the maturity of the convertible bonds. With a positive probability it will be smaller than if a call is enforced. In other words, the convertible bond possibly protects the stockholders against a wealth reduction.

(5) The value of a callable convertible bond is discontinuous for the critical firm values at which the optimal call policy changes. The convertible bond value jumps upwards (downwards), if the debt holders suffer (benefit) from the call. Due to this jump caused by a call, the value of a *callable* convertible bond can be *higher* than an otherwise identical but non-callable convertible bond.

(6) Contrary to the convertible bond value, the stock value is continuous and (weakly) increasing in the firm value. Surprisingly, a higher call price can result in a higher stock value. This is the case if a call results in a partial conversion.

We made a number of simplifying assumptions to reveal in a transparent way the determinants why it can be optimal to call convertible bonds too late or too early. Most of these assumptions can be modified without changing the basic results at the expense that the results become more opaque.

(1) The structure of the optimal call and conversion decisions does not change if more call dates are considered.

(2) If the notice period has a positive length an additional step is necessary. In this step, conditional on a call the discounted expected value of the conversion dependent convertible bond values at the end of the notice period must be computed. Of course, the conversion decision is not known at the call date.

(3) If the convertible bonds are coupon bearing, each coupon date represents a possible default date. If, as typically in the European markets, coupon dates coincide with call induced repayment dates, the structure of the problem does not change. However, the coupon payments affect the size of the wealth transfers. If the additional bonds bear a positive coupon, in general additional default dates exist.

Again, this extension primarily affects the size of the critical firm values but not their structure.

(4) If the stock pays a positive dividend, voluntary conversions are possible. If the dividend is proportional to the firm value at the dividend date, it can be shown that a full conversion is optimal for sufficiently large firm values. Thus, for very high firm values a complete conversion occurs independent of the call decision of the firm.

(5) If the call price C differs from the face value of the convertible bonds, the difference $D_0\left(1,0,V_0^-\right) - D_0\left(0,0,V_0^-\right)$ in the debt values is affected. This call-dependent term was important to study the optimal call policy. If $C > F$ ($C < F$) the debt holders suffer more (less) from a call as the firm value V_0^- drops more (less) than in the case $C = F$.

(6) If the additional bonds mature before the convertible bonds ($TD < T$), a conversion after TD causes no wealth transfer between stock and additional bondholders. Therefore, between TD and T, the traditional IBS strategy is optimal. In call dates before TD, the debt holders can only suffer but not gain from a call because the only possibility for a wealth transfer affecting the debt holders is a call with an at least partial redemption of the convertible bonds.

(7) Finally, we will discuss a modification of our priority rule. If the additional bonds are senior and the convertible bonds are junior, the optimal conversion policy in $t = 0$ and $t = T$ will not change but the modified priority rule affects the value of the convertible bonds. This effect will change the critical firm values $V_C^{(1)}$ and $V_C^{(2)}$ but not the structure of the optimal call policy.

To demonstrate the effect of a reverse priority rule, we consider the example presented in Figure 5. A call is optimal under the old priority rule for firm values larger than $V_C^{(1)} = 71,065$. For a default the firm value has to decrease by more than 80% with a risk-neutral probability considerably below 1%. These values give an indication for the minor role of the priority rule. In fact, if additional bonds are senior, the critical firm values do not numerically differ from those presented in Section 4.3. These findings are not in line with a statement by Ingersoll (1977a, p. 299) that if a firm is levered with senior debt as well as convertible subordinated debentures it will be still optimal to call the convertible bonds as soon it forces a full conversion.

It will be an interesting study to test the predictions of this paper empirically. This project is left for further research.

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